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Abstract Monomorphisms Between Big Subgroups of Some Groups of Type B_2 in Characteristic 2

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Abstract monomorphisms of sufficiently "full" subgroups of some groups of type B_2 in characteristic 2 into other groups of type B_2 (over any characteristic) are considered and described. The result is that they are induced by a semi-algebraic special isogeny. The proof uses the method of O. T. O'Meara as well as some other technique.

0. INTRODUCTION

0.1. We show below that standard results, see, e.g. [2, 14], hold for monomorphisms between big subgroups of some groups of type B_2. The main part (Section 4) of the proof is straightforward although tedious, and does not use any new ideas or techniques. In this part of the paper we use identification of adjoint groups of type B_2 defined over a field k with orthogonal groups of a quadratic vector space V over k of dimension 5 and defect 1 (recall: char $k = 2$). Our assumption is that V contains a regular hyperplane with Arf invariant 0. In algebraic group terms it can be formulated as follows: there exists a one-dimensional k-torus T in G such that the semi-simple part of $Z_G(T)$ is a group of type A_1 generated by root subgroups corresponding to long roots. This assumption is satisfied if G is split over k or is anisotropic and split over a separable quadratic extension of k.

0.2. Let us now state our result. Let k be an infinite field of characteristic 2 and let V be a quadratic vector space over k, of defect 1, dim $V = 5$, and $G = SO(V)$. A subgroup H of $G(k)$ is called *big* if for every singular k-torus $T \subset G$ the intersection $T \cap H$ is infinite. Let k' be another field, let V' be a quadratic space over k', of defect at most 1, dim $V' = 5$ and $G' = SO(V')$.

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0.3. MAIN THEOREM. *Suppose that V contains a regular hyperplane with Arf invariant* 0. *Let $\alpha: H \to G'(k')$ be a monomorphism. Then there exists a unique field homomorphism $\varphi: k \to k'$ and a unique special k'-isogeny $\beta: {}^{\varphi}G \to G'$ such that $\alpha(h) = \beta(\varphi^0(h))$ for $h \in H$.*

Here, in the statement and notations, we follow [2]. In particular ${}^{\varphi}G$ stands for $\varphi(k)$-group obtained from G by base change φ and φ^0: $G(k) \to {}^{\varphi}G(\varphi k))$ is the corresponding homomorphism of the group of k-rational points. An isogeny, we recall, is special if its differential is not identically zero (cf. [2, Sect. 3]).

0.4. The history of the study of homomorphisms between orthogonal groups is roughly as follows. J. Dieudonné [7, Chap. IV] studied automorphisms of and isomorphisms between isotropic orthogonal groups. Next M. J. Wonenburger [18–21] extended his results on automorphisms to full orthogonal groups of degree $\geqslant 7$ in characteristic $\neq 2$, but the groups were permitted to be anisotropic. Then in 1968, O. T. O'Meara [14] introduced his method of residual spaces. He and his school used this method to study isomorphisms and automorphisms of many subgroups of orthogonal groups (including some congruence subgroups), see [8–15].

But O'Meara's result did not include characteristic 2. Connors [3–6] extended the theory to characteristic 2 using a blend of O'Meara's method and the method of involutions. For isotropic groups a recent paper [11] by A. Hahn deals with the isomorphism problem of "large" subgroups. However, in all these papers only isomorphisms are considered.

The only paper which treats arbitrary homomorphisms is Borel and Tits [2]. They consider (among others) isotropic orthogonal groups and subgroups therein containing all groups $U(k)$, U a unipotent radical of a k-parabolic.

0.5. We proceed as follows. We denote by Y the set of singular k-subtori of G. The set Y is a disjoint union of two subsets Y_l and Y_s where Y_l (resp., Y_s) consists of $T \in Y$ such that the roots of G lying in $Z_G(T)$ are long (resp., short). As we noted before our assumption on G is that $Y_l \neq \phi$. If $\alpha: H \to G'(k')$ is a monomorphism one shows that α gives rise to an injection $\bar{\alpha}: Y \to Y'$ where Y' is the corresponding object for G'. It turns out that either $\bar{\alpha}(Y_l') \subseteq Y_l'$ or $\bar{\alpha}(Y_l) \subseteq Y_s'$. We consider first the former case. Then we identify elements of Y_s with regular subplanes of V and for regular subplanes belonging to a fixed (but arbitrary) regular hyperplane N of V with Arf invariant 0 we can describe in group terms when two regular planes belong to a three-dimensional subspace of N. This permits one, using standard arguments, to define a monomorphism of the projective space of N into a similar thing in V'. We apply the fundamental theorem of projective geometry to this map and then show that the field homomorphisms corresponding to different N's coincide with some $\varphi: k \to k'$ and the semi-linear maps agree. This permits us to glue these semi-linear maps together and obtain a semi-linear map $V \to V'$. Then we show that it gives rise to an isomorphism $\beta: {}^{\varphi}G \to G'$.

Next we consider the case $\bar{\alpha}(Y_l) \subseteq Y_s'$. To deal with this case we take a third group G'' of type B_2 split and adjoint over k' and consider a special inseparable isogeny ν: $G' \to G''$ which inverts lengths of roots. We apply our previous result to $\nu \circ \alpha$ and then using certain pull-back procedure and results of our paper [17], we establish the Main Theorem in full generality. This place is the only one where we use a trick.

0.6. Conventions and notations. We denote by $M(k)$ the set of rational points of an algebraic variety M defined over a field k. Sometimes we write M_k to indicate that M is defined over k. If φ: $k \to k'$ is a field homomorphism we denote by ${}^{\varphi}M$ the algebraic variety over k' obtained from M by base change φ. We denote by φ^0 the corresponding mapping φ^0: $M(k) \to {}^{\varphi}M(k')$. By an algebraic group we understand a reduced affine group scheme of finite type over a field. For an algebraic (resp., abstract) group G and an algebraic (resp., abstract) subset $M \subset G$ we denote by $Z_G(M)$, $N_G(M)$, D^iM, $C(M)$ the algebraic (resp., abstract) subgroup of G which is respectively the centralizer of M in G, the normalizer of M is G, i-th derived group of M, the center of M. If G is as above and M_1 ,..., M_n are algebraic (resp., abstract) subgroups of G then $\langle M_1 ,..., M_n\rangle$ denotes the algebraic (resp., abstract) group generated by the groups M_i, $i = 1, 2,..., n$. If G is an algebraic group then G^0 denotes its connected component. If H is an algebraic group and T is a subtorus of H then $\Sigma(H, T)$ denotes the set of roots of T in H. For a commutative group S we denote by S^m, $m \in \mathbf{Z}$, the group of m-th powers of elements of S. Eor a set X we denote by $|X|$ its cardinality. For a field k we denote by $\bar{k}$ its algebraic closure. An element x of an algebraic group G is called here regular if it is semi-simple and regular. For a map β: $X \to Y$ and $Z \subseteq X$ we denote by $\beta \mid Z$ or $\beta|_Z$ the restriction of β to Z.

If V is a vector space over k and M is its subspace we denote by $\mathbf{P}V$ and $\mathbf{P}M$ the combinatorial projective spaces associated to V and M and by $\mathbf{P}_V^{\dim V-1}$ and $\mathbf{P}_M^{\dim M-1}$ the algebraic varieties which are projective spaces associated to V and M. We always assume that canonically $\mathbf{P}M \subseteq \mathbf{P}V$ and $\mathbf{P}_M^{\dim M-1} \subseteq \mathbf{P}_V^{\dim V-1}$. If V and M are as above we call M line, plane, hyperplane of V if respectively $\dim M = 1$, $\dim M = 2$, $\dim M = \dim V - 1$.

0.7. *Recollections about quadratic spaces and orthogonal groups.* General references: [7, 13, 16]. We recall first that all adjoint groups of type B_2 defined over a field k can be realized as orthgonal groups of an appropriate 5-dimensional quadratic space over k.

If V is a vector space over a field k with a quadratic form f we denote by F the associated bilinear form. For a subspace $M \subseteq V$ we denote by $M^\perp$ the orthogonal complement of M with respect to F. If M, $N \subseteq V$ are two subspaces we write $M \perp N$ for the space $M + N$ if the sum is direct and $F(M, N) = 0$. If M is a subspace of V we say that M is *regular* if $\dim(M \cap M^\perp) \leqslant 1$ and $f(M \cap M^\perp) \neq 0$ in the case $M \cap M^\perp \neq 0$. The integer $\dim(M \cap M^\perp)$ is called the *defect* of M. If char $k = 2$ then for a regular even-dimensional subspace

$M \subseteq V$ we denote by $q(M)$ the Arf invariant of M. A hyperplane $N \subset V$ is called here *admissible* if it is regular and $q(N) = 0$.

We consider regular 5-dimensional space V over a field k with char $k = 2$. In this case dim $M - \dim(M \cap M^\perp)$ is even for any $M \subseteq V$. In particular, defect $V = 1$. If M is a plane, $M \subseteq V$, then either $M \cap M^\perp = 0$ or $M \subset M^\perp$. If M is a 3-dimensional subspace of V then either $M \subset M^\perp$ or dim $M \cap M^\perp = 1$. If M is a hyperplane then either $M \cap M^\perp = 0$, or dim $M \cap M^\perp = 2$ (the case $M \subset M^\perp$ would imply that V is not regular). If N is a regular hyperplane in V and $M \subseteq N$, dim $M = 3$, then M is regular since the case $M \subseteq M^\perp$ would contradict regularity of N. In particular, if N, $\tilde{N}$, $N \neq \tilde{N}$, are two regular hyperplanes then $N \cap \tilde{N}$ is a regular 3-dimensional space.

We write $SO(M)^0$ although $SO(M) = O(M)$ if char $k = 2$ and $SO(M) = SO(M)^0$ if char $k \neq 2$.

0.8. To conclude this introduction I record my sincere thanks to Edward A. Connors with whom I discussed the subject of this paper on several occasions and whose help and advice were essential for me.

1. Subgroup of $SO(V)$

1.1. Let $G = SO(V)$. We consider the set Y of one-dimensional k-subtori T of G satisfying two properties: $D^2Z_G(T) \neq 1$ and $T = C(Z_G(T))^0$. If $\tilde{T}$ is a maximal torus, containing T, then $\Sigma(Z_G(T), \tilde{T})$ is a pair of opposite roots, say $\{\pm a\}$. We write $T \in Y_s$ if a is a short root and $T \in Y_l$ if a is a long root. Thus $Y = Y_s \cup Y_l$, $Y_s \cap Y_l = \phi$. We say that $T \in Y_s$ (resp. $T \in Y_l$) is of short (resp. long) type. If T is a k-subtorus of G then the representation of T on V is completely reducible so that $V = V^T \perp V_T$ where $V^T = \{v \in V \mid Tv = v\}$. If $T \in Y_s$ then V_T is a regular plane and $T = SO(V_T)^0$, so that T can be identified with the group of rotations in the plane V_T.

1.2. Similarly, if M is a regular plane then the group $SO(M)^0$ can be considered as a subgroup of G (since $O(M)$ is generated by orthogonal transvections which it contains). Then $SO(M)^0$ is a one-dimensional k-torus. For every subspace N of V such that $F(N, N) \neq 0$ we denote by G_N the subgroup of G generated by groups $SO(M)^0$ when M runs over regular subplanes of N. In particular, we have $G_M = SO(M)^0$ when M is a regular plane.

1.3. Lemma. *If N is a regular subspace of V then $G_N = SO(N)^0$.*

Proof. We remarked that it is true if dim $N = 2$. If dim $N = 3$ then $SO(N)^0$ is of type A_1 and therefore it is generated by any pair of distinct tori it contains, so $G_N = SO(N)^0$. If dim $N = 4$, then we take two regular 3-dimensional subspaces M_1, M_2 in N such that $M_1 \neq M_2$. Then $G_N \supset G_{M_1}, G_{M_2}$ and clearly, $\langle G_{M_1}, G_{M_2} \rangle = SO(N)^0$.

1.4. Proposition. *The correspondences $T \in Y_s \to V_T$ and $M \to G_M$, where M is a regular plane, are inverse to one another and they establish a bijection between Y_s and the set of regular subplanes of V.*

Proof. We actually mentioned all necessary ingredients of the proof. But let us repeat them. Let $T \in Y_s$. Then $V = V^T \oplus V_T$ is a direct orthogonal sum whence it follows that V_T is a regular plane. Then $T_{V_T} \subseteq SO(V_T)^0$ and since $\dim T = \dim SO(V_T) = 1$, we have $T = SO(V_T)^0$. Now G_M is a one-dimensional k-torus and $Z_G(G_M) \supseteq G_{M^\perp}$. Since $M^\perp$ is regular, $G_{M^\perp}$ is of type A_1 whence $G_M \in Y$. It can be checked that G_M is of type Y_s.

1.5. Lemma. *The correspondence $N \to G_N$ defines a bijection between regular 3-dimensional subspaces of V and k-subgroups of type A_1 of G whose maximal k-tori belong to Y_s.*

Proof. We know that G_N is of type A_1 and its maximal k-tori belong to Y_s. Let G_1 be a k-subgroup of G of type A_1 whose maximal k-tori belong to Y_s and let $T_1, T_2, T_1 \neq T_2$, be two such tori. Then $G_1 = \langle T_1, T_2 \rangle$ and on the other hand by definition $\langle T_1, T_2 \rangle = G_N$ where $N = V_{T_1} + V_{T_2}$. If $\dim N > 3$ then G_N is not of type A_1, a contradiction.

1.6. *Remark.* There are two different kinds of regular 3-dimensional subspaces N of V. If $N \supset V^\perp$ then $Z_G(G_N) \supseteq G_{N^\perp}$ is again of type A_1 and $G_N \subseteq Z_G(D^r Z_G(G_N))$ for all $r \geqslant 0$. If $N \cap V^\perp = 0$ then $F(N^\perp, N^\perp) = 0$ and therefore $D^2 Z_G(G_N) = 1$. This later fact is one of major obstructions to a standard proof.

1.7. Lemma. *Let N be a regular 4-dimensional subspace of V. Then $q(N) = 0$ if and only if G_N contains a subtorus $T \in Y_l$. If G_N contains such a subtorus then $G_N = Z_G(T)^0 \cdot Z_G(DZ_G(T))^0$.*

Proof. We know (by [16, Corollary 5]) that $q(N) = 0$ iff G_N is an almost direct product of two k-groups of type A_1. Suppose first that $q(N) = 0$ then $G_N = G_1 \cdot G_2$, $[G_1, G_2] = 0$. Take for T a maximal k-subtorus of G_1. Then $Z_G(T)^0 = T \cdot G_2$, $DZ_G(T)^0 = G_2$, $Z_G(DZ_G(T)) = G_1$. This shows that $T \in Y$ and that the last assertion of 1.7 holds. By 1.4, $T \notin Y_s$, i.e., $T \in Y_l$. If, on the contrary, G_N contains a torus $T \in Y_l$, then $Z_G(T)^0 = Z_{G_N}(T)^0$ and we easily derive that $G_N = G_1 \cdot G_2$ whence $q(N) = 0$.

1.8. Lemma. *Let $T \in Y_l$ and set $G_1 = Z_G(T)^0 \cdot Z_G(DZ_G(T))^0$. Then there exists a regular hyperplane N of V with $q(N) = 0$ such that $G_1 = G_N$.*

Proof. The group G_1 is a group of type $A_1 \times A_1$. It preserves the space V_T so that $G_1 \subseteq SO(V_T)^0$. But $SO(V_T)$ is of type $A_1 \times A_1$ since $\dim V_T = 4$. Therefore $G_1 = SO(V_T)$. Now $q(V_T) = 0$ by 1.7. So our assertion holds with $N = V_T$.

1.9. LEMMA. *Let N be a regular hyperplane of V and $M \subset N$ be a regular 3-dimensional subspace of N. Then G_M is a maximal connected algebraic subgroup of G_N having the properties: G_M contains a subtorus from Y_s and G_M does not contain regular elements of G.*

Proof. Clearly G_M does have the above properties. Suppose that it is not maximal. Then there exists $T \in Y_s$ such that $T \notin G_M$ but $\langle G_M, T\rangle$ does not contain regular elements. This is absurd since $\langle G_M, T\rangle = G_N$.

1.10. LEMMA. *Let N be a regular hyperplane of V and let G_1 be a maximal connected algebraic subgroup of G_N which has the properties stated in 1.9. Then there exists a regular 3-dimensional subspace M of N such that $G_1 = G_M$.*

Proof. Let R be the set of $T \in Y_s$ such that $T \subset G_1$. Then $G_1 \supseteq \langle T, T \in R\rangle$. Therefore $G_1 \supseteq G_M$ where $M = \sum_{T \in R} V_T$. If dim $M = 2$ then G_1 is not maximal. If dim $M = 4$ then G_1 does contain regular elements. So dim $M = 3$. Since $M \supseteq V_T$ we see that the defect of M is at most 1 so M is regular. Thus $G_1 \supseteq G_M$ with M regular, dim $M = 3$. Since such a G_M is maximal connected in G_N, we have $G_1 = G_M$, as asserted.

2. SPECIES OF TORI

2.1. DEFINITION. Let $T, T' \in Y$. We write $T \sim T'$ (and say that T and T' *are of the same species*) if there exists a sequence $T_1 = T, T_2, \ldots, T_n = T'$ with $T_i \in Y$ such that $[T_i, T_{i+1}] \neq 1$ and $\langle T_i, T_{i+1}\rangle$ does not contain regular elements (for $i = 1, 2, \ldots, n-1$).

2.2. LEMMA. *If $T_1 \in Y_s$, $T_2 \in Y_l$ then $\langle T_1, T_2\rangle$ does contain regular elements.*

Proof. Let $G_1 = \langle T_1, T_2\rangle$. Let T be a maximal torus of G_1. Then T_i is conjugate in G_1 to a subtorus $\tilde{T}_i$ of T. Since T_1 is not conjugate to T_2 in G we see that $\tilde{T}_1 \neq \tilde{T}_2$ whence dim $\tilde{T}_1 \cdot \tilde{T}_2 = 2$, i.e., dim $T = 2$, i.e., G_1 contains regular elements.

2.3. COROLLARY. *If $T_1 \in T_s$, $T_2 \in Y_l$ then T_1 and T_2 are not of the same species.*

2.4. LEMMA. *If $T, T' \in T_s$ then $T \sim T'$.*

Proof. Let $M = V_T$, $M' = V_{T'}$. If $F(M, M') \neq 0$ we choose $v \in M$, $v' \in M'$ with $F(v, v') \neq 0$ and set $M_1 = M$, $M_2 = kv + kv'$, $M_3 = M'$. The plane M_2 is regular by construction and M_1, M_3 are regular by Proposition 1.4. Hence $T_i = G_{M_i} \in Y_s$ and $T_1 = T$, $T_3 = T'$ by 1.4. Therefore $N_1 = M_1 + M_2$ and $N_2 = M_2 + M_3$ are regular 3-dimensional spaces. Therefore $G_{N_1} =$

$\langle T_1, T_2\rangle$ and $G_{N_2} = \langle T_2, T_3\rangle$ do not contains regular elements (cf. Lemma 1.5). Thus T_1, T_2, T_3 is the desired sequence.

If $F(M, M') = 0$ then choose a regular plane M'' such that $F(M, M'') \neq 0$ and $F(M'', M') \neq 0$. The argument of the previous paragraph applies to pairs M, M'' and M'', M'. Since $\sim$ is a transitive relation, our assertion follows.

2.5. Corollary. *Let $T \in Y_s$ and $T' \in Y$. Then $T \sim T'$ if and only if $T' \in Y_s$.*

Proof. Combine 2.4 and 2.3.

2.6. *Remark.* Thus we know that tori of short type form a single species and tori of different length type belong to different species. We do not claim however, that tori of long type form a single species.

3. Big Subgroups

3.1. Definition. A subgroup H of $G(k)$ is called *big* if $T \cap H$ is infinite for any $T \in Y$.

3.2. *Remark.* The above says that $T \cap H$ is Zariski dense in T. This permits one to identify Y with a set of subgroups of H. We shall use this identification as a matter of convenience.

3.3. To express in abstract group terms different statements of Sections 1 and 2 we need simple density arguments which we record below.

Lemma. *Let G be an algebraic k-group and let $G_1, G_2, ..., G_m$ be closed algebraic k-subgroups of G. Let H be a dense subgroup of G and let H_i be a dense subgroup of G_i.*

(i) *If H_1 is normal in H then G_1 is normal in G and $H/H \cap G_1$ is dense in G/G_1;*

(ii) $Z_G(H_1) = Z_G(G_1)$ *and* $Z_H(H_1) = Z_G(G_1) \cap H$;

(iii) D^iH *is dense in* D^iG;

(iv) *if G is connected and reductive then D^iH is dense in DG for all $i \geqslant 1$; in particular, if G is not a torus then $D^iH \neq 1$ for $i \geqslant 1$;*

(v) *if G_1 is connected and reductive then $Z_H(D^iH_1) = Z_H(DH_1) = H \cap Z_G(DG_1)$;*

(vi) $\langle H_1, ..., H_m\rangle$ *is dense in* $\langle G_1, ..., G_m\rangle$;

(vii) *if G is not unipotent, H contains a regular element of G of arbitrary large order.*

Proof. In (i) it is clear that G_1 is normal in G. Let $\bar{G}$ be the closure of HG_1/G_1 in G/G_1. Then the preimage of $\bar{G}$ contains H whence $\bar{G} = G/G_1$. The assertion

(ii) is clear. Now (iii) follows from (i) if we take G_1 to be the closure of $H_1 = DH$. Next, (iv) holds because $DG = D^iG$ for $i > 1$ and because of (iii). Now (v) is the combination of (ii) and (iv). To prove (vi) note that $H_{i_1} \times H_{i_2} \times \cdots \times H_{i_n}$ is dense in $G_{i_1} \times G_{i_2} \times \cdots \times G_{i_n}$ whence the image of the first product in G is dense in the image of the second. Finally, (vii) holds since the set of regular elements of order $\geqslant n_0$ is open in G and non-empty for any $n_0 \in \mathbf{Z}$.

3.4. In our case, when $G = SO(V)$ we can slightly strengthen the assertion 3.3(vii). We need

DEFINITION. An element $x \in G$ is called *very regular* if x^{2^5} is a regular element of G.

Remark. The element x^{2^5} is automatically semi-simple since $u^{2^5} = 1$ for every unipotent $u \in G$.

3.5. LEMMA. *Let M be either V or a regular subspace of V. Let H be a big subgroup of $G(k)$.*

(i) *$H \cap G_M$ is dense in G_M;*

(ii) *$x \in H$ is very regular if and only if $Z_H(x^{2^5})$ is commutative.*

Proof. By definition of bigness, $H \cap G_N$ is dense in G_N for every regular plane $N \subset M$. Thus (i) follows from 3.3(vi). The assertion (ii) is not clear only in one direction. To prove it in that direction, assume that $Z_H(x^{2^5})$ is commutative. If x^{2^5} is not regular, then $DZ_G(x^{2^5})$ is a group of type A_1. Therefore $T = C(Z_G(x^{2^5}))$ belongs to Y. Hence $Z_H(x^{2^5}) \supseteq Z_H(T) \supset DZ_H(T)$. But the group $DZ_H(T)$ contains all groups $T' \cap H$ where T' runs over maximal k-tori of $DZ_G(T)$. Since these tori belong to Y and generate $DZ_G(T)$ we see that $DZ_H(T)$ is dense in $DZ_G(T)$ and therefore highly non-commutative (cf. 3.3(iv)).

3.5.1. *Remark.* We implicitly used the fact that semi-simple elements in G are not of order 2^i, $i \geqslant 1$. Therefore the centralizers of semi-simple elements cannot have a maximal torus as a subgroup of finite index n, $n \geqslant 2$ (because such situation occurs only for elements of order 2.)

3.6. LEMMA. *Let H' be a subgroup of H and let G' be the closure of H' in G. The group G' contains regular elements of G if and only if H' contains very regular elements.*

Proof. The group G' contains very regular elements if and only if it contains regular ones. If it contains regular elements then the set of very regular elements is open in G' (it consists of regular elements whose order is $> n_0$ for some $n_0 \in \mathbf{Z}$). Therefore H' intersects non-trivially with this set.

3.7. Proposition. *A subgroup $S \subseteq H$ belongs to Y if and only if it has the following properties*

(i) *S is commutative, infinite and of unbounded period;*

(ii) *$S = C(Z_H(S^m))$ and $D^i(Z_H(S^m)) \neq 1$ for any $i \geqslant 0$ and any $m \geqslant 1$.*

Proof. If $S \in Y$ then $S = T \cap H$ for some $T \in Y$ and so it is infinite and commutative by condition. It is of unbounded period since the only torsion subgroup of $T(\bar{k}) \simeq \bar{k}^*$ is the group of roots of unity. Now the group $DZ_G(T)$ is of type A_1 and therefore it is generated by k-tori which it contains. These k-tori belong to Y and therefore have dense intersection with H. Using 3.5(vi) we see that $H \cap DZ_G(T)$ is dense in $DZ_G(T)$ whence (ii).

Suppose on the contrary that S satisfies (i) and (ii). Let G_m be the Zariski closure of S^m in G. The group G_m is commutative and therefore $G_m{}^0 = T_m \times U_m$ with T_m a torus and U_m unipotent. Clearly $G_1{}^m \supseteq G_m$ whence it follows that $G_m = T_m$ if $2^5 \mid m$. By (ii) we have $DZ_G(T_m)$ is non-commutative and therefore of type A_1. This would show that $S \in Y$ once we show that T_m is defined over k. This follows from [1, 10.3] since S^m consists of semi-simple elements and $S^m \subseteq G(k)$.

3.8. Proposition. *Let S, $S' \in Y$. The following are equivalent*

(i) *There exists a sequence $S = S_1, S_2, \ldots, S_n = S'$ with $S_i \in Y$ such that $[S_i, S_{i+1}] \neq 1$ and $\langle S_i, S_{i+1} \rangle$ does not contain very regular elements for $i = 1, \ldots, n-1$;*

(ii) *the closures of S and S' are of the same species.*

Proof. If (ii) holds then (i) follows from 2.5. Assume that (i) holds Let T_i be the closure of S_i. Then $T_i \in Y$ and $[T_i, T_{i+1}] \neq 1$. Now $\langle T_i, T_{i+1} \rangle$ does not contain regular elements because of 3.6 and therefore $T_1 \sim T_n$ as required.

3.9. Lemma. *Let H_1 be a subgroup of H. The following are equivalent*

(i) *There exists $S \in Y_l$, $S \subset H_1$ such that $H_1 = N_H(H_2)$ where $H_2 = Z_H(S) \cdot Z_H(DZ_H(S))$*

(ii) *There exists an admissible hyperplane N in V such that $N_G(G_N) \cap H = H_1$.*

Proof. Clearly (i) implies (ii) (take into account 1.7, 1.8). To prove that (ii) implies (i) we note that by 1.7 there exists $T \in Y_l$ such that $G_N = Z_G(T)^0 \cdot Z_G(DZ_G(T))^0$. Thus H_2 is normal in $N_G(G_N) \cap H$ whence our assertion.

3.10. Lemma. *Let N be an admissible hyperplane of V. Let H_1 be a subgroup of $N_G(G_N) \cap H$. The following are equivalent:*

(i) *$H_1 = N_{N_G(G_N)\cap H}(G_M)$ for some regular 3-dimensional subspace M of N;*

(ii) *H_1 is a maximal subgroup of $N_G(G_N) \cap H$ which does not contain very regular elements but does contain a subgroup $S \in Y_s$.*

Proof. It follows from 3.6 and 1.9.

3.11. Now we sum up our achievements. To do this recall that we can identify Y_s with the set of regular planes in V (by 1.4). Now using 3.9 we can, given an admissible hyperplane $N \subset V$ decide whether a regular plane V_S, $S \in Y_s$, belongs to N. Then using 3.10 we can decide whether two regular subplanes of N belong to one regular three-dimensional space. All this is done using abstract group terms.

3.12. *Remark.* Note that we have no means to describe what is a plane or a three-dimensional space in inner terms of the group $H \cap N_G(G_N)$.

4. Monomorphisms Preserving Lengths of Roots

Let k' be another field and let G' be an adjoint algebraic group of type B_2 over k'. We identify G' with $SO(V')$ where V' is a quadratic vector space over k' of dimension 5 with quadratic form f' and associated bilinear form F'. Let Y', Y'_s, Y'_l be the sets of tori defined for G' as in 1.1. Note that 1.1–1.5 as well as Section 2 have sense and hold in all characteristics.

From now on we make the following

4.1. Assumption. V contains an admissible hyperplane (or, equivalently, $Y_l \neq \phi$).

Let H be a big subgroup of $G(k)$ and let α: $H \to G'(k')$ be a monomorphism. For $S \in Y$ let $H_1(S) = DZ_H(S)$ and $H_2(S) = Z_H(DZ_H(S))$. Both these groups are dense in groups of type A_1 and therefore for any subgroup $\tilde{H}_i(S) \subseteq H_i(S)$ of finite index we have $D^m\tilde{H}_i(S) \neq 1$ for $i = 1, 2$ and any $m \geqslant 0$. Denote by $\bar{\alpha}(S)$, $G'_2(S)$, $G'_1(S)$ the connected components of the Zariski closures in G' of $\alpha(S)$, $\alpha(H_1(S)$, $\alpha(H_2(S))$. These groups are closures of subgroups of finite index in $\alpha(S)$, $\alpha(H_1(S))$, $\alpha(H_2(S))$ respectively.

4.2. Lemma. *For $S \in Y$ the groups $G'_1(S)$ and $G'_2(S)$ are commuting subgroups of type A_1 in G'.*

Proof. We have $D^mG'_i(S) \neq 1$ for any $m \geqslant 0$. Therefore the groups $G'_i(S)$ are not solvable. But any pair of non-solvable commuting subgroups of a group of type B_2 is a pair of subgroups of type A_1, whence our assertion.

4.3. LEMMA. (i) char $k' = 2$

(ii) *There exist two groups* $S, \bar{S} \in Y$ *having different length type such that*

$$\bar{\alpha}(S) \in Y'_s, \qquad \bar{\alpha}(\bar{S}) \in Y'_l.$$

Proof. Let N be an admissible hyperplane of V. Let $T \in Y_l$ be such that $G_N = Z_G(T) \cdot Z_G(DZ_G(T))^0$. Let $S = T \cap H$. Then the groups $G'_i(S)$ are closures of subgroups $\alpha(\tilde{H}_i(S))$ of finite index in $\alpha(H_i(S))$. Recall that every unipotent in $Z_G(T)^0$ and $Z_G(DZ_G(T))^0$ is of order 2. Consider in G'_i the set X'_i of semi-simple elements of order > 2. Then X'_i is open and therefore $\alpha(\tilde{H}_i(S)) \cap X'_i \neq \varnothing$. Let $\alpha(x) \in \alpha(\tilde{H}_1(S)) \cap X'_1$. Then x is semi-simple. Therefore $Z_G(x)$ is defined over k (cf. [1, 3.10]). Hence $T_1 = C(Z_G(x))$ belongs to Y_l (we have $D^m Z_G(x) \neq 1$ for all $m \geqslant 0$ since $Z_G(x) \supseteq DZ_G(T)$). We have, of course, that $\bar{\alpha}(T_1 \cap H) \in Y'$ since $\alpha(x)$ is regular in $G'_1(S)$. Let $S_1 = T_1 \cap H$. Then $G'_i(S) = G'_i(S_1)$ for $i = 1, 2$. Set $T'_1 = \bar{\alpha}(S_1)$.

Let now M be a regular 3-dimensional subspace of N and $\tilde{H} = H \cap G_M$. Let $\tilde{G}'$ be the connected component of the closure of $\alpha(\tilde{H})$ in G'. The group $\tilde{G}'$ is not solvable, since $\tilde{H}$ is dense in a group of type A_1. Next, since we have for any $x \in \tilde{H}$ that $D^m Z_H(x^2) \neq 1$ for $m \geqslant 0$, it follows that $\tilde{G}'$ does not contain regular elements of G'. On the other hand, by density argument there exists $y \in \tilde{H}$, $y^2 \neq 1$, such that $\alpha(y)$ is regular in $\tilde{G}'$. Then y is semi-simple (in G). Let $T'_2 = C(Z_{G'}(\alpha(y)))$, $T_2 = C(Z_G(y)) = Z_{G_M}(y)$, $S_2 = T_2 \cap H$. By construction, $T_2 \in Y_s$. If T'_2 is of the same length type as T'_1 then $G'_1(S) \cdot G'_2(S) = Z_{G'}(T'_2)^0 \cdot Z_G(DZ_{G'}(T'_2))^0$ and $Z_{G'}(T'_2) \subset G'_1(S) \cdot G'_2(S)$ whence $Z_H(y) \subset H \cap G_N$, whence $Z_{G_M}(y) \subset G_N$, which is false. Thus T'_2 is of the different length type. This proves part (ii) of our Lemma.

Suppose that char $k \neq 2$. Then the only groups of type $A_1 \times A_1$ in G' are of the form $G_{N'}$ for some regular hyperplane $N' \subset V'$. Thus $G'_1(S) \cdot G'_2(S) = G_{N'}$. Thus $T'_1 \in Y'_l$ whence $T'_2 \in Y'_s$. Then $DZ_{G'}(T'_2) = G'_{M'}$ for some 3-dimensional subspace $M' \subset V'$ and we have $D^2 Z_{G'}(G'_{M'}) = 1$ (in characteristic $\neq 2$). But for S_2 we have: $D^m Z_H(DZ_H(S_2)) \neq 1$ which is a contradiction with the inclusion $Z_{G'}(G'_{M'}) \supset \alpha(Z_H(DZ_H(S_2)))$. This proves (i).

4.4. LEMMA. (i) *If* $S \in Y$ *then* $\bar{\alpha}(S) \in Y'$,

(ii) *If* $S_1, S_2 \in Y$ *and* $\bar{\alpha}(S_1) = \bar{\alpha}(S_2)$ *then* $S_1 = S_2$,

(iii) *if* $S_1, S_2 \in Y$ *are of the same species then so are* $\bar{\alpha}(S_1), \bar{\alpha}(S_2)$,

(iv) *if* $S_1 \in Y_l$, $S_2 \in Y_s$ *then* $\bar{\alpha}(S_1)$ *and* $\bar{\alpha}(S_2)$ *are of different length type* (*and, in particular, of different species*).

Proof. Take $x \in S$ such that $x^{2^5} \neq 1$. Then $\alpha(x^{2^5}) = (\alpha(x))^{2^5}$ is a semi-simple element of $G'(k')$. Since by 4.2 we have that $D^m Z_{G'}(\alpha(x^{2^5})) \neq 1$ for $m \geqslant 0$ it follows that $T' = C(Z_{G'}(\alpha(x^{2^5})))$ is a one-dimensional k'-subtorus of G' and $T' \in Y'$. Clearly, $T' \supset \alpha(S)$. This proves (i).

Now (ii) is clear since if $\bar{\alpha}(S_1) = \bar{\alpha}(S_2)$ then $[S_1, S_2] = 1$ and $Z_H(S_1) = Z_H(S_2)$ whence $S_1 = S_2$.

To prove (iii) it is sufficient to show that if $S_3, S_4 \in Y$ are such that $[S_3, S_4] \neq 1$ and $\langle S_3, S_4 \rangle$ does not contain very regular elements of H then $[\bar{\alpha}(S_3), \bar{\alpha}(S_4)] \neq 1$ (it is evident) and $\langle \bar{\alpha}(S_3), \bar{\alpha}(S_4) \rangle$ does not contain regular elements of G' (cf., 2.1 and 3.8). Since $\langle S_3, S_4 \rangle$ does not contain very regular elements we have that $D^m Z_H(x^{2^5}) \neq 1$ for all $x \in \langle S_3, S_4 \rangle$ and all $m \geqslant 0$. But if $\langle \bar{\alpha}(S_3), \bar{\alpha}(S_4) \rangle$ contains a regular element then their set is open and therefore, by 3.3(vi), (vii), there would exist $x \in \langle S_3, S_4 \rangle$ such that $Z_{G'}((\alpha(x))^{2^5})$ is commutative. This is a contradiction.

Let us prove (iv). It was actually proved in the proof of 4.3(ii). Namely, it was shown there that there exists $\bar{S} \in Y_s$, $\bar{S} \subset Z_G(S_1)^0 \cdot Z_G(DZ_G(S_1))^0$ such that $\bar{\alpha}(\bar{S})$ and $\bar{\alpha}(S_1)$ are of different length type. By 4.4(iii) above we have that $\bar{\alpha}(\bar{S})$ and $\bar{\alpha}(S_2)$ are of the same length type whence our assertion.

4.5. Corollary. *The map $\bar{\alpha}$: $Y \to Y'$ is injective and we have either $\bar{\alpha}(Y_s) \subseteq Y'_s$ and $\bar{\alpha}(Y_l) \subseteq Y'_l$ or $\bar{\alpha}(Y_s) \subseteq Y'_l$ and $\bar{\alpha}(Y_l) \subseteq Y'_s$.*

4.6. We assume for the remainder of this Section that $\bar{\alpha}(Y_s) \subseteq Y'_s$ and $\bar{\alpha}(Y_l) \subseteq Y'_l$. The case $\bar{\alpha}(Y_s) \subseteq Y'_l$ will be considered in the next Section. Our use of the assumption $\bar{\alpha}(Y_s) \subseteq Y'_s$ is based on the fact that Y_s (resp., Y'_s) can be identified with the set of regular planes in V (resp., in V'), cf. Proposition 1.4. Thus $\bar{\alpha}$ can be considered as a map, denoted $\bar{\alpha}_2$, from the set of regular planes of V to the set of regular planes of V'.

4.7. Lemma. *Let N be an admissible hyperplane in V and let M be a regular hyperplane in N. Then there exists an admissible hyperplane N' in V' and a regular hyperplane M' of N' such that the connected components of the closures of $\alpha(G_N \cap H)$ and $\alpha(G_M \cap H)$ are $G'_{N'}$ and $G'_{M'}$ respectively.*

Proof. The fact that there exists such an N' follows from the assumption $\bar{\alpha}(Y_l) \subseteq Y'_l$, Lemma 4.2 and, for $q(N') = 0$, from Lemma 4.2 and [16]. The existence of M' was implicitly established before. Let us repeat the corresponding argument. Let $\tilde{G}'$ be the connected component of the closure of $\alpha(G_M \cap H)$. Then $\tilde{G}'$ is not solvable, it contains tori from Y_s and does not contain regular elements of G' (otherwise $G_M \cap H$ contains very regular elements of H). The only possibility for a group with these properties is that $\tilde{G}' = G'_{M'}$ for a regular hyperplane M' of N'.

4.8. Corollary. (i) *$\bar{\alpha}$ determines a map $\bar{\alpha}_4$ of the set of admissible hyperplanes of V to a similar set in V';*

(ii) *$\bar{\alpha}$ determines a map $\bar{\alpha}_{3,N}$ of the set of regular 3-dimensional subspaces of an admissible hyperplane N to the set of regular hyperplanes of $\bar{\alpha}_4(N)$.*

4.9. LEMMA. (i) $\bar{\alpha}_4$ *is an injection,*

(ii) $\bar{\alpha}_{3,N}$ *is an injection,*

(iii) *if N is an admissible hyperplane of V and M_2, M_3 are regular 2- and 3-dimensional subspaces of N then $M_2 \subset M_3$ if and only if $\bar{\alpha}_2(M_2) \subset \bar{\alpha}_{3,N}(M_3)$.*

Proof. Let us start with (iii). If $M_2 \subset M_3$ then $G_{M_2} \subset G_{M_3}$ and $G_{M_2} \cap H \subset G_{M_3} \cap H$, hence $\alpha(G_{M_2} \cap H) \subset \alpha(G_{M_3} \cap H)$. Therefore, if $M_2 \subset M_3$ then $\bar{\alpha}_2(M_2) \subset \bar{\alpha}_{3,N}(M_3)$. Suppose now that $\bar{\alpha}_2(M_2) \subset \bar{\alpha}_{3,N}(M_3)$ but $M_2 \not\subset M_3$. Suppose that there exists a regular hyperplane $\tilde{M}_3$ in N such that $\bar{\alpha}_{3,N}(\tilde{M}_3) \neq \bar{\alpha}_{3,N}(M_3)$ and $\tilde{M}_3 \supset M_2$. Then $\bar{\alpha}_2(M_2) = \bar{\alpha}_{3,N}(\tilde{M}_3) \cap \bar{\alpha}_{3,N}(M_3)$ whence $M_2 \subseteq M_3 \cap \tilde{M}_3$, as required. Therefore we have to assume that for every regular hyperplane $\tilde{M}_3$ of N with $\tilde{M}_3 \supset M_2$ we have $\bar{\alpha}_{3,N}(\tilde{M}_3) = \bar{\alpha}_{3,N}(M_3)$.

Let us now show that, for any regular plane $\tilde{M}_2 \subset N$, one has $\bar{\alpha}_2(\tilde{M}_2) \subset \bar{\alpha}_{3,N}(M_3)$. This is clear if $\tilde{M}_2 \subset M_3$. So assume that $\tilde{M}_2 = kx + ky$ with $x \notin M_3$. Then the preceeding argument shows that $\bar{\alpha}_{3,N}(M_2 + kx) = \bar{\alpha}_{3,N}(M_3)$. Let $\bar{M}_2$ be a regular plane in $M_2 + kx$ containing x, $\bar{M}_2 \neq \tilde{M}_2$. Then we have $\bar{\alpha}_2(\bar{M}_2) \subset \bar{\alpha}_{3,N}(M_3)$ but $\bar{M}_2 \not\subset M_3$. Therefore (again by the preceeding argument) $\bar{\alpha}_{3,N}(\bar{M}_2 + ky) = \bar{\alpha}_{3,N}(M_3)$ whence $\bar{\alpha}_2(\tilde{M}_2) \subset \bar{\alpha}_{3,N}(M_3)$. In group terms it says that $\bar{\alpha}(G_{\tilde{M}_2} \cap H) \subset G_{\bar{\alpha}_{3,N}(M_3)}$ for all regular planes $\tilde{M}_2 \subset N$. Therefore $G'_{\bar{\alpha}_4(N)}$ normalizes $G'_{\bar{\alpha}_{3,N}(M_3)}$, a contradiction.

Let us prove (ii). If M_3, $\tilde{M}_3$ are two regular hyperplanes in N such that $\bar{\alpha}_{3,N}(M_3) = \bar{\alpha}_{3,N}(\tilde{M}_3)$ but $M_3 \neq \tilde{M}_3$ then there exists a regular plane $M_2 \subset M_3$ such that $M_2 \not\subset \tilde{M}_3$ and we can apply (iii) to get a contradiction.

Let us prove (i). If $N, \tilde{N}$, are two admissible hyperplanes and $\alpha_4(N) = \alpha_4(\tilde{N})$ let us write almost direct products $G_N = G_1 \cdot G_2$, $G_{\tilde{N}} = \tilde{G}_1 \cdot \tilde{G}_2$, $G'_{\alpha_4(N)} = G'_1 \cdot G'_2$ with G_1, G_2, $\tilde{G}_1$, $\tilde{G}_2$, G'_1, G'_2 of type A_1. Let $T \subset \tilde{G}_2$, $T \in Y_l$. Then $\tilde{G}_1 = DZ_G(T)$ whence $G'_1 = DZ_{G'}(\bar{\alpha}(T \cap H))$ whence $G_1 \subseteq DZ_G(T)$. Thus $G_1 = \tilde{G}_1$. Similarly, $G_2 = \tilde{G}_2$, i.e. $G_N = G_{\tilde{N}}$, i.e. $N = \tilde{N}$. This concludes the proof of the lemma.

4.10. LEMMA. *Let M_1, M_2, M_3 be three pairwise different regular subplanes of an admissible hyperplane N of V. Then* $\dim(M_1 \cap M_2 \cap M_3) = 1$ *if and only if* $\dim \bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) \cap \alpha_2(M_3) = 1$.

Proof. Suppose first that $\dim(M_1 \cap M_2 \cap M_3) = 1$. Then by 4.4(ii) and 4.9(iii) we get that $\bar{\alpha}_2(M_i) \cap \bar{\alpha}_2(M_j)$ is a line for $i \neq j$. Let $R = M_2 + M_3$. Then R is a regular hyperplane of N. Suppose first that $M_1 \not\subset R$. Then $\dim(M_1 \cap R) = 1$ and by 4.9(iii) $\dim(\bar{\alpha}_2(M_1) \cap \bar{\alpha}_{3,N}(R)) = 1$. Therefore $\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_i) = \bar{\alpha}_2(M_1) \cap \bar{\alpha}_{3,N}(R)$. Therefore $\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) \cap \bar{\alpha}_2(M_3) = \bar{\alpha}_2(M_1) \cap \bar{\alpha}_{3,N}(R)$ is a line as required. Now consider the case $M_1 \subset R$. Then there exists a regular plane M_4 such that $M_4 \subset N$, $M_4 \supset M_1 \cap M_2 \cap M_3$ but $M_4 \not\subset M_1 + M_2 = M_1 + M_3 = M_2 + M_3$. Apply the preceeding result to the triples (M_4, M_2, M_3), (M_4, M_1, M_3), (M_4, M_1, M_2) and obtain the result in the full generality.

Suppose now that $\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) \cap \bar{\alpha}_2(M_3)$ is a line. Then $R'_{ij} = \bar{\alpha}_2(M_i) + \bar{\alpha}_2(M_j)$, $i \neq j$, $i, j = 1, 2, 3$, is a hyperplane of $N' = \bar{\alpha}_4(N)$. By 4.9(iii) we see that $R_{ij} = M_i + M_j$ is a hyperplane of N. Therefore $L_{ij} = M_i \cap M_j$, $i \neq j$, is a line. If two of L_{ij} coincide then they coincide with the third one. So assume that all L_{ij} are distinct. Set $R = L_{12} + L_{13} + L_{23} = M_1 + M_2 + M_3$. Then by 4.9(iii) we have $\bar{\alpha}_2(M_i) \subset \bar{\alpha}_{3,N}(R) = R'_{12} = R'_{13}$. Now find a regular plane M_4 in N such that $M_4 \supset L_{12}$ but $M_4 \not\subset R$. Then $\bar{\alpha}_2(M_4) \not\subset R'_{12}$ but $M'_1 \cap M'_2 \cap M'_3 \cap M'_4$ is a line. We have $L_{12} = M_1 \cap M_2 \cap M_4$. On the other hand since $\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_3) \cap \bar{\alpha}_2(M_4)$ is a line but $M_4 \not\subset R = M_1 + M_3$, we must have (to avoid a contradiction with the above argument) that $M_1 \cap M_3 \cap M_4$ is a line. But then $M_1 \cap M_2 \cap M_4 = M_1 \cap M_4 = M_1 \cap M_3 \cap M_4$ whence $L_{12} = M_1 \cap M_4 = M_1 \cap M_2 \cap M_3 \cap M_4$, a contradiction with the assumptions.

4.11. Corollary. *For regular planes $M_1, M_2 \subset N$, $\dim(M_1 + M_2) = 3$, $\dim N = 4$, N admissible, set $\bar{\alpha}_{1,N}(M_1 \cap M_2) = \bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2)$. Then $\bar{\alpha}_{1,N}$ is a map of the set of lines of N into the set of lines of N'.*

4.12. Lemma. *$\bar{\alpha}_{1,N}\colon \mathbf{P}N \to \mathbf{P}N'$ is an injection.*

Proof. Suppose $\bar{\alpha}_{1,N}(M_1 \cap M_2) = \bar{\alpha}_{1,N}(M_3 \cap M_4)$ with $M_i \subset N$, $\dim(M_1 + M_2) = \dim(M_3 + M_4) = 3$. Then it means that $\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) = \bar{\alpha}_2(M_3) \cap \bar{\alpha}_2(M_4)$ whence $\dim(\bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) \cap \bar{\alpha}_2(M_3) \cap \bar{\alpha}_2(M_4)) = 1$. Now 4.10 implies that $M_1 \cap M_2 = M_3 \cap M_4$.

4.13. *Remark.* We are trying to reduce everything to the fundamental theorem of projective geometry. It would be probably easier (although less explicit) to use Dieudonné's Theorem [7, III, Sect. 2].

4.14. Lemma. *Let L be a line and let R be a regular three-dimensional subspace of V. Suppose there exist two admissible hyperplanes N, $\tilde{N}$ of V such that R, $L \subset N \cap \tilde{N}$. Then $\bar{\alpha}_{3,N}(R) = \bar{\alpha}_{3,\tilde{N}}(R)$, $\bar{\alpha}_{1,N}(L) = \bar{\alpha}_{1,\tilde{N}}(L)$* (where equality is considered as equality of subspaces of V' or, the same, of $\bar{\alpha}_4(N)$).

Proof. The assertion for $\bar{\alpha}_{3,N}$ is clear because of 4.9(ii) and (iii). To prove the assertion for $\bar{\alpha}_{1,N}$ recall that $R_1 = N \cap \tilde{N}$ is a regular hyperplane of N and $\tilde{N}$. Take in R_1 two regular planes, say M_1, M_2 such that $L = M_1 \cap M_2$. Then by definition $\bar{\alpha}_{1,N}(L) = \bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_2) = \bar{\alpha}_{1,\tilde{N}}(L)$, as required.

4.15. Corollary. *There exist well defined injections*

(i) $\bar{\alpha}_3$: *of the set of regular 3-dimensional subspaces of V contained in admissible hyperplanes of V into the set of 3-dimensional subspaces of V';*

(ii) $\bar{\alpha}_1$: *of the set of lines contained in admissible hyperplanes of V into the set of lines of V'.*

4.16. LEMMA. *Let N be an admissible hyperplane of V and let L_1, L_2, L_3 be three lines of N. Then* $\dim(L_1 + L_2 + L_3) = 2$ *if and only if* $\dim(\bar{\alpha}_1(L_1) + \bar{\alpha}_1(L_2) + \bar{\alpha}_2(L_3)) = 2$.

Proof. Assume first that $M = L_1 + L_2 + L_3$ is a plane. Take two regular hyperplanes R_1, R_2 of N such that $R_1 \cap R_2 = M$. For line L_i we find in R_j a regular plane M_{ij}, $i = 1, 2, 3, j = 1, 2$, such that $M_{ij} \supset L_i$, $j = 1, 2$. Then $\bar{\alpha}_1(L_i) = \bar{\alpha}_2(M_{i1}) \cap \bar{\alpha}_2(M_{i2})$ whence $\bar{\alpha}_1(L_i) \subset \bar{\alpha}_3(R_1) \cap \bar{\alpha}_3(R_2)$. Since $\bar{\alpha}_3(R_1) \neq \bar{\alpha}_3(R_2)$ (by 4.9(ii)) we have $\dim(\bar{\alpha}_3(R_1) \cap \bar{\alpha}_3(R_2)) = 2$, whence our assertion.

Suppose now that $M' = \bar{\alpha}_1(L_1) + \bar{\alpha}_1(L_2) + \alpha_2(L_3)$ is a plane, but $R = L_1 + L_2 + L_3$ has dimension 3. Find a line $L \subset N$; $L \not\subset R$, such that the planes $M_i = L_i + L$ were regular and $R_1 = L_1 + L_2 + L \neq R_2 = L_1 + L_3 + L$. Then $\bar{\alpha}_2(M_1)$, $\bar{\alpha}_2(M_2) \subset \bar{\alpha}_3(R_1)$ and $\bar{\alpha}_3(M_1)$, $\bar{\alpha}_2(M_3) \subset \bar{\alpha}_3(R_2)$. Therefore $\bar{\alpha}_1(L_1)$, $\bar{\alpha}_1(L_2) \subset \bar{\alpha}_3(R_1)$ and $\bar{\alpha}_1(L_1)$, $\bar{\alpha}_1(L_3) \subset \bar{\alpha}_3(R_2)$. Hence (since $\bar{\alpha}_1(L_i)$ are pairwise distinct by 4.12) we have $M' \subset \bar{\alpha}_3(R_1) \cap \bar{\alpha}_3(R_2)$. Since $R_1 \cap R_2 = M_1$, we have $M' = \bar{\alpha}_2(M_1)$. Let $\tilde{M}_i$ be a regular plane, $\tilde{M}_i \subset N$, $\tilde{M}_i \supset L_i$, $\tilde{M}_i \neq M_1$. Then for any $i = 1, 2, 3$ the planes $\bar{\alpha}_2(M_1)$, $\bar{\alpha}_2(\tilde{M}_i)$ are contained in a regular 3-dimensional space whence the same holds for M_1, $\tilde{M}_i$ (by 4.9(iii)), i.e. $\tilde{L}_i = M_1 \cap \tilde{M}_i$ is a line. But we have $\bar{\alpha}_1(L_i) = \bar{\alpha}_2(M_1) \cap \bar{\alpha}_2(M_i) = \bar{\alpha}_1(\tilde{L}_i)$. Thus $L_i = \tilde{L}_i$ (by 4.12) and therefore $L_1 + L_2 + L_2 \subset M_1$, as required.

4.17. COROLLARY. *For an admissible hyperplane N of V we have*: $\bar{\alpha}_1 \colon \mathbf{P}N \to \mathbf{P}\bar{\alpha}_4(N)$ *is an imbedding of projective spaces respecting incidence structure.*

4.18. COROLLARY. *For an admissible hyperplane N of V there exist a unique field homolorphism $\varphi_N \colon k \to k'$ and a unique k'-isomorphism of algebraic varieties $\bar{\beta}_N \colon {}^{\varphi_N}\mathbf{P}_N{}^3 \to \mathbf{P}^3_{\bar{\alpha}_4(N)}$ such that $\bar{\alpha}_1 = \beta_N \circ \varphi_N{}^0$ on $\mathbf{P}_N{}^3(k) = \mathbf{P}N$.*

4.19. LEMMA. *If N, $\tilde{N}$ are two admissible hyperplanes of V then $\varphi_N = \varphi_{\tilde{N}}$.*

Proof. We have $\dim(N \cap \tilde{N}) = 3$ and by 4.14 $\bar{\alpha}_1$ induces a map of projective spaces $\mathbf{P}(N \cap \tilde{N}) \to \mathbf{P}(\bar{\alpha}_3(N \cap \tilde{N}))$. By unicity statement in the fundamental theorem of projective geometry, we have $\varphi_N = \varphi_{\tilde{N}}$ as claimed (and also $\bar{\beta}_N \mid {}^{\varphi}\mathbf{P}^2_{N \cap \tilde{N}} = \bar{\beta}_{\tilde{N}} \mid {}^{\varphi}\mathbf{P}^2_{N \cap \tilde{N}}$).

4.20. *Notation.* $\varphi = \varphi_N$.

4.21. Let N and $\tilde{N}$ be as in 4.12. Let us define $\bar{\beta}_{N,\tilde{N}} \colon {}^{\varphi}\mathbf{P}_V{}^4 \to \mathbf{P}_{V'}{}^4$ in the following way. Let $\tilde{\beta}_N \colon {}^{\varphi}N \to \bar{\alpha}_4(N)$ be a linear map whose projectivisation is $\bar{\beta}_N$. It is defined up to a multiplication by an element from $\varphi(k)$. Since by the unicity statement in the fundamental theorem of projective geometry $\bar{\beta}_N \mid {}^{\varphi}\mathbf{P}^2_{N \cap \tilde{N}} = \bar{\beta}_{\tilde{N}} \mid {}^{\varphi}\mathbf{P}^2_{N \cap \tilde{N}}$ we infer that there exists a constant $\lambda_{N,\tilde{N}} \in \varphi(k)$ such that $\tilde{\beta}_N \mid {}^{\varphi}(N \cap \tilde{N}) = (\lambda_{N,\tilde{N}} \cdot \tilde{\beta}_{\tilde{N}}) \mid {}^{\varphi}(N \cap \tilde{N})$. We define a linear map $\tilde{\beta}_{N,\tilde{N}} \colon {}^{\varphi}V \to V'$ by setting $\tilde{\beta}_{N,\tilde{N}} \mid {}^{\varphi}N = \tilde{\beta}_N$ and $\tilde{\beta}_{N,\tilde{N}} \mid {}^{\varphi}\tilde{N} = \lambda_{N,\tilde{N}} \cdot \tilde{\beta}_{\tilde{N}}$.

Clearly, this is a well-defined map. Let $\bar{\beta}_{N,\tilde{N}}$ be its projectivisation. Then $\bar{\beta}_{N,\tilde{N}}$ does not depend on choices of $\tilde{\beta}_N$ and $\tilde{\beta}_{\tilde{N}}$.

4.22. *Remark.* $\bar{\beta}_{N,\tilde{N}} = \bar{\beta}_{\tilde{N},N}$.

4.23. LEMMA. $\bar{\beta}_{N,\tilde{N}}$ does not depend on the choice of N, $\tilde{N}$.

Proof. Let $\bar{N}$ be a third admissible hyperplane in V. Consider first the general case: $\bar{N} \cap N \neq \bar{N} \cap \tilde{N}$. We have $\tilde{\beta}_N \mid {}^{\varphi}(N \cap \bar{N}) = \lambda_{N,\bar{N}} \cdot \tilde{\beta}_{\bar{N}} \mid {}^{\varphi}(N \cap \bar{N})$ and $\tilde{\beta}_{\tilde{N}} \mid {}^{\varphi}(\tilde{N} \cap \bar{N}) = \lambda_{\tilde{N},\bar{N}} \cdot \tilde{\beta}_{\bar{N}} \mid {}^{\varphi}(\tilde{N} \cap \bar{N})$. Since $\dim(N \cap \tilde{N} \cap \bar{N}) = 2 > 0$ it follows from $\tilde{\beta}_N \mid {}^{\varphi}(N \cap \tilde{N}) = \lambda_{N,\tilde{N}} \cdot \tilde{\beta}_{\tilde{N}} \mid {}^{\varphi}(N \cap \tilde{N})$ that $\lambda_{N,\bar{N}} = \lambda_{N,\tilde{N}} \cdot \lambda_{\tilde{N},\bar{N}}$. Therefore $\bar{\beta}_{N,\bar{N}} = \bar{\beta}_{\tilde{N},\bar{N}}$, whence our assertion.

If $\bar{N} \cap N = \tilde{N} \cap \bar{N}$ then we choose a fourth admissible hyperplane N_4 in V such that the intersections $\bar{N} \cap N_4$, $\tilde{N} \cap N_4$, $N \cap N_4$ are pairwise distinct (it is possible because of the density and general position argument.) Applying the above argument to our three pairs we obtain the desired result.

4.24. *Notation.* The common value of $\bar{\beta}_{N,\tilde{N}}$ from 4.21 will be denoted $\bar{\beta}$. We denote by β the induced k'-homomorphism ${}^{\varphi}PGL(V) \to PGL(V')$.

4.25. COROLLARY. *For any admissible hyperplane N of V we have $\alpha(h) = \beta(\varphi^0(h))$ for $h \in G_N \cap H$.*

4.26. COROLLARY. $\alpha(h) = \beta(\varphi^0(h))$ *for any $h \in H$.*

Proof. By 4.25 α coincides with $\beta \cdot \varphi^0$ on the subgroup $\bar{H}$ of H generated by all groups $G_N \cap H$, N an admissible hyperplane of V. Since $\bar{H}$ is normal we can apply [2, 8.10] and get the result. (The homomorphism μ of [2, 8.10] is trivial since $C(G') = \{1\}$.)

5. MONOMORPHISMS INVERTING LENGTHS OF ROOTS

Now we are going to complete the proof of our Main Theorem by treating the case $\bar{\alpha}(Y_s) \subseteq Y_l'$. To do this we do not need to represent our group as an orthogonal group. Rather we use structure results of algebraic group theory.

5.1. *Assumption.* $\bar{\alpha}(Y_s) \subseteq Y_l'$.

5.2. Let $T_1 \in Y_l$ and let T_2 be a maximal k-subtorus of $DZ_G(T_1)$ (then $T_2 \in Y_l$). Then $T = T_1 \cdot T_2$ is a maximal k-subtorus of G. We know that $T_i' = \bar{\alpha}(T_i) \in Y_s$ whence $T' = T_1' \cdot T_2'$ is a maximal subtorus of G'. Let G'' be a split adjoint group of type B_2 over k'. Then (using [2, 3.8, 3.9]) we can construct a special k'-isogeny ν: $G' \to G''$ (denoted β_0 in [2, 3.8]) using T' as T of [2, 3.8]. We set $T'' = \nu(T')$. Since ν inverts lengths of roots we see that $\alpha_1 = \nu \circ \alpha$: $H \to G''(k')$ preserves lengths of roots: $\alpha_1(Y_s) \subseteq Y_s''$. Therefore 4.26 is applicable and there exist a unique field homomorphism $\varphi_1: k \to k'$ and a unique k'-isomorphism $\beta_1: {}^{\varphi_1}G \to G''$ such that $\alpha_1(h) = \beta_1(\varphi_1{}^0(h))$ for $h \in H$.

5.3. Since root systems $\Sigma(G, T)$, $\Sigma(G', T')$, $\Sigma(G'', T'')$ are isomorphic we use the same notations for elements of these three sets distinguishing them only by an appropriate number of primes. If $a \in \Sigma(G, T)$ we denote by U_a the root subgroup corresponding to root a. We can assume (cf. [2, 3.1]) that $\beta_1^*: \Sigma'' \to \Sigma$ and $\nu^*: Q\Sigma'' \to Q\Sigma'$ are such that $\beta_1^*(a'') = a$ and $\nu^*(a'') = \lambda(a) \cdot \rho(a)'$ where $\rho: \Sigma \to \Sigma$ is a bijection permuting lengths of roots and $\lambda(a) = 1$ (resp., 2) if a is long (resp., short). By our choice of $T = T_1 \cdot T_2$, $T_i \in Y_i$, the groups $G(a) = \langle U_a, U_{-a}\rangle$, a long, are defined over k. We have for them: $\alpha(G(a) \cap H) \subseteq G'(\rho(a))$ (because of our choice of notation and of the actions of β^*, ν^*).

5.4. To continue we need to recall the existence of a projective plane structure (cf. [17]) on the set $S(\bar{G})$ of connected one-dimensional subgroups of an algebraic group G of type A_1. The lines of this structure are of two kinds: parabolic lines and involutorial lines. A parabolic line is the set of all connected one-dimensional subgroups contained in a Borel subgroup. An involutorial line is the set of all connected one-dimensional subgroups such that all elements of these subgroups are inverted by an involutive automorphism depending only on the line. Since $\beta_1(a) = \beta_1 \mid G(a): {}^{\varphi_1}G(a) \to G''(a'')$ is an isomorphism for $a \in \Sigma(G, T)$, a long, and since $\mathrm{Fr} = \nu \mid G'(\rho(a)') \to G''(a'')$ is the Frobenius map for a long, we see that $\beta(a)$ induces an isomorphism $\bar{\beta}: {}^{\varphi}S(G(a)) \to S(G''(a''))$ and ν induces the Frobenius map $\mathrm{Fr}: S(G'(\beta(a)')) \to S(G''(a''))$. It follows from $\alpha_1 = \beta_1 \circ \varphi_1{}^0$, $\alpha_1 = \nu \circ \alpha$ that α maps subgroups of $G(a) \cap H$ which belong to a line into subgroups of $G'(\rho(a)')$ which belong to a line and that any set of subgroups which do not belong to one line is mapped into the set of subgroups which do not belong to a line. Thus ([17], 4.3.3) is applicable to the map $\alpha_{a,T}: G(a) \cap H \to G'(\rho(a)')$. Therefore $\alpha_{a,T}(h) = \beta_{a,T}(\varphi^0_{a,T}(h))$ for $h \in H \cap G(a)$ where $\varphi_{a,T}: k \to k'$ is a field homomorphism and $\beta_{a,T}: {}^{\varphi_{a,T}}G(a) \to G'(\rho(a)')$ is an isomorphism, with both $\beta_{a,T}$ and $\varphi_{a,T}$ unique. Now since $\nu: G'(\rho(a)') \to G''(a'')$ coincides with Frobenius we have $\varphi_1 = \mathrm{Fr} \circ \varphi_{a,T}$. Therefore $\varphi_{a,T}$ does not depend on a maximal k-torus T (chosen, however, according to the procedure described in the beginning of 5.2) and on a long root $a \in \Sigma(G, T)$. Thus we denote $\varphi_{a,T} = \varphi$. We have $\varphi_1 = \mathrm{Fr} \circ \varphi$. Since $\varphi_1 = \mathrm{Fr} \circ \varphi$ we can consider $\beta_2 = \beta_1 \circ \mathrm{Fr}$ and we have $\alpha_1 = \beta_2 \circ \varphi^0$ on H. Now we again appeal to [2, 3.8] and find a k-isogeny $\beta: {}^{\varphi}G \to G'$ such that $\beta_2 = \nu \circ \beta$. Since β_1 is an isomorphism and ν is a special isogeny we see that β is a special isogeny. Since ν is a monomorphism on the set of points it follows that $\alpha(h) = \beta(\varphi^0(h))$ for $h \in H$. This completes the proof of the Main Theorem.

5.5. *Remarks.* (i) The use of the results of [17] does not seem to be essential. They are used only to establish that there exists a p-th root of φ_1. However, I do not see any way to get it directly.

(ii) One can prove the same result assuming that only G' (and not G) is adjoint. It does not change anything, however, since in characteristic 2 the central isogenies are injective on the set of points.

(iii) One can prove that G' is actually defined over $\varphi(k)$ and that β is a $\varphi(k)$-isogeny.

(iv) Although we used the assumption that $H \cap T$ is infinite for $T \in Y$ it is quite possible that one can get through with a weaker assumption: $H \cap T$ contains more than two elements.

Note added in proof. The 3-dimensional subspaces used in the proofs of Lemmas 4.9, 4.10, 4.14 might be non-regular. The situation can be corrected by "general position" argument followed by a specialization argument as 4.23 or the first paragraph of 4.10. Another way to correct the situation is to drop the adjective "regular" in relation to 3-dimensional spaces in all proofs and statements of Section 4 starting with Lemma 4.8. To justify this one should add

4.7. BIS LEMMA. *Lemma 4.7 holds also when M is not regular.*

Proof. The group $\tilde{G}'$ (in notation of 4.7) has the properties: it is solvable, noncommutative, contains tori from Y_s, does not contain regular elements of G'. Therefore, $\tilde{G}' = G'_{M'}$ for some non-regular hyperplane M' of N'.

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