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On Abstract Homomorphisms of Anisotropic Algebraic Groups over Real-Closed Fields

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Abstract homomorphisms with Zariski-dense image of the group of rational points of anisotropic almost simple algebraic groups over real closed fields into other almost simple algebraic groups are described. The result states, in particular, that the kernel of such homomorphism is a congruence subgroup of the original group.

0. INTRODUCTION

0.1. We shall consider fields more general than the real closed fields. Namely, we consider fields satisfying

R1. The field $K = k((-1)^{1/2})$ has no quadratic extensions.

R2. The quadratic form $u^2 + x^2 + y^2 + z^2$ does not represent zero.

The examples of such fields are

(a) real closed fields,

(b) let K be any subfield of $\mathbf{C}$ closed under quadratic extensions, let k be the fixed subfield of the complex conjugation.

We consider algebraic semi-simple groups which are defined and anisotropic over k and split over K. We call such groups *admissible*. Using results and ideas of [17], we obtain for such groups the simplest structural results (cf. Section 3).

As a parallel development we prove in Appendix 1 a Klingenberg's version of the fundamental theorem of projective geometry and apply it in Section 4 to prove our main result for groups of type A_1. Then in Sections 5 and 6 we combine results of Sections 4 and 3 and obtain:

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0.2. MAIN THEOREM. *Suppose that G is absolutely almost simple, simply connected, and admissible. Suppose that k' is another field and G' an absolutely almost simple k'-group. Let $\alpha: G(k) \to G'(k')$ be a homomorphism with dense image. Then there exist*

(i) *a unique place $\varphi: k \to k'$; let A be its valuation ring;*

(ii) *a unique structure G_A of a semi-simple group scheme over A on G; we have $G_A(A) = G(k)$ in this structure.*

(iii) *a unique structure of an algebraic $\varphi(A)$-group on G';*

(iv) *a unique central $\varphi(A)$-isogeny β of algebraic $\varphi(A)$-groups, $\beta: {}^{\varphi}G_A = G_A \otimes_{\varphi} \varphi(A) \to G'_{\varphi(A)}$*

such that

$$\alpha(g) = \beta(\varphi^0(g)) \qquad \textit{for} \quad g \in G(k).$$

Here ${}^{\varphi}G_A$ can be roughly pictured as the reduction of G_A modulo the maximal ideal of A and φ^0 can be pictured as the homomorphism of reduction considered on points $G_A(A)$ of G_A.

0.3. The most interesting features of the above Theorem are (i) and (ii). They say that given a homomorphism we can find a subring in our field such that the group is actually defined over this subring and such that our homomorphism is actually the reduction modulo the maximal ideal of our ring. Of course, this generality may be vacuous if our subring A is k itself. But if k is real closed and non-archimedean, then we can take the subring A of k as the set of elements which are not infinitely big in some fixed order. Then any admissible group will be automatically defined over A (cf., B. Pollack [12] for a similar but stronger statement or E. Artin [1, Chap. V, Sect. 3] for an example) and one can construct non-trivial homomorphisms by taking the reduction modulo the ideal of infinitely small elements in A. Thus (i) and (ii) are not vacuous. It can be shown, however, that $A = k$ if the kernel of α is in the center of $G(k)$. On the other hand, it is very probable that our subring A is always the ring of not infinitely large elements of k (recall that it is known that our field is uniquely orderable, e.g. S. Lang [9, bottom of p. 381]). So the degree of generality of our Theorem may be quite restricted.

0.4. The next question which naturally arises is: why do we assume that the field is real closed or something like real closed. To explain this we have to say a few words about the proof. The proof for arbitrary groups G is based on the validity of the theorem in the particular case of groups of type A_1. In this case we proved in [18] that G' must be also of type A_1. In the case when G and G' are both of type A_1 we have a projective plane structure on the set of connected one-dimensional subgroups of G and G' (cf. [18]) and α induces a "map" $\bar{\alpha}: \mathbf{P}^2(k) \to \mathbf{P}^2(k')$ which preserves incidence. In general, this "map" need not be everywhere defined (this is the reason for putting quotation marks around "map"). Namely, if $\alpha(H(k)) = \{1\}$ for some connected one-dimensional k-subgroup H of G then $\bar{\alpha}$ is not defined in the corresponding point of $\mathbf{P}^2(k)$. However, in the case we consider in the present paper. $\bar{\alpha}$ is everywhere defined since all connected one-dimensional k-subgroups of G are conjugate in $G(k)$ (cf. 2.1 below).

0.5. For an (everywhere defined) map $\bar{\alpha}: \mathbf{P}^2(k) \to \mathbf{P}^2(k')$ which preserves incidence an analog of the fundamental theorem of projective geometry holds in the form (cf. [8]): there exist a valuation subring A of k, an A-lattice M in the underlying space V of $\mathbf{P}^2(k)$, a ring homomorphism $\varphi: A \to k'$, a map of A-modules $\bar{\beta}: M \to V'$ (V' is the underlying space of $\mathbf{P}^2(k')$ made into an A-modules via φ) such that $\bar{\alpha}$ is the composition of $\bar{\beta}$ with natural projections.

0.6. There are, however, other reasons why our ideas may fail or lead to unmanageable considerations for other fields. One of them is that generally we can define our projective plane structure in abstract group terms only for adjoint groups of type A_1. For our fields k the difference between adjoint and simply connected groups of type A_1 is negligible (cf. 2.4 and 4.4.4 below).

0.7. Another much more conceptual difficulty is the following. Suppose we have a generalization of the Fundamental Theorem of Projective geometry to maps which are not everywhere defined (cf. 0.8(ii) below for a more precise statement). Then we get a local subring A of k and we have to define some A-structure on G (which may not even be a group scheme structure). I do not know of any way to do that. In our case we get a structure of a semi-simple group scheme on G and that is easy to describe.

0.8. The above discussion suggests the following questions.

(i) Can one study order-preserving maps between Tits buildings? (For split Tits buildings it seems to be easy.) Can such maps be used to describe Bruhat–Tits buildings?

(ii) Can one use the full generality of Klingenberg's paper [8] to get the the results of the following sort: Let G be an absolutely simple algebraic group defined over a field k and let H be a big (in some appropriate sense) subgroup of $G(k)$. Let G' be another absolutely simple algebraic group over a field k' and let $\alpha: H \to G'(k')$ be a homomorphism with a dense image. Then there exists a local subring $A \subseteq k$ and a structure of a group scheme over A on G such that $H \subset G(A)$ and α is a composition of a homomorphism $\varphi: A \to k'$ with a special isogeny a homomorphism $\varphi: A \to k'$ with a special k'-isogeny $\beta: G_{\varphi(A)\cdot k'} \to G_{k'}$.

(iii) Can one combine the methods of this paper with methods of J. Tits [16] to get a description of all homomorphisms with dense image of our groups into perfect algebraic groups? If so, one will get a solution of a kind of congruence subgroup problem.

(iv) Let D be a central division algebra over k, D^1 the set of elements of D of norm 1, $\tilde{D}$ the subring of D generated by D^1. Is it true that D^1 has normal subgroups if and only if $\tilde{D} \neq D$? Do these normal subgroups correspond to ideals of $\tilde{D}$ (or even of $\tilde{D} \cap$ (center of D))?

0.9. We refer to [2, 6, 11, 15] for a historical survey and to [10, pp. 255–259] for a complete bibliography of papers on homomorphisms between algebraic groups covering years 1928–1975.

0.10. *Conventions and notations*. For a field k satisfying R1, R2 we set $K = k((-1)^{1/2})$. The Galois group of K over k acts on $a = b + (-1)^{1/2}c$, $b, c \in k$, by $\bar{a} = b - (-1)^{1/2}c$. Then $N_{K/k}(a) = a \cdot \bar{a}$, $K^1 = \{a \in K \mid N_{K/k}(a) = 1\}$. Sometimes we write N for $N_{K/k}$. By D we denote the division algebra of quaternions over k; we have $D = \{\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) \mid a, b \in K\}$. The norm map $Nrd\colon D \to k$ is defined by $Nrd\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) = N(a) + N(b)$. Then $D^1 = \{d \in D \mid Nrd(d) = 1\}$.

For an unramified extension A_1 of a ring A we denote by $\mathrm{Gal}(A_1/A)$ the Galois group of A_1 over A. If we have an A-scheme X we indicate it by writing X_A. The points of X in an overring $A_1 \supset A$ are denoted $X(A_1) = X_A(A_1)$. If $\varphi\colon A \to B$ is a homomorphism of rings we denote by ${}^{\varphi}X_A$ the B-scheme obtained from X by the base change φ and $\varphi^0\colon X(A) \to {}^{\varphi}X_A(B)$ is the corresponding mapping on points.

For an algebraic (resp., abstract) group G and algebraic (resp., abstract) subset $M \subseteq G$ we denote by $Z_G(M)$, $N_G(M)$ the algebraic (resp., abstract) subgroup of G which is the centralizer or, respectively, the normalizer of M in G. For an algebraic group G we denote by G^0 its connected component and by $R_u(G)$ its unipotent radical.

If G is an algebraic group and T a subtorus of G then $\Sigma(G, T)$ is the set of roots of T in G (or in the Lie algebra of G). For a subset $\Sigma_1 \subset \Sigma(G, T)$ we denote by $G(\Sigma_1)$ the subgroup of G generated by root subgroups of T in G corresponding to roots from $\pm\Sigma_1$. In particular, if T is a maximal torus and $a \in \Sigma(G, T)$ then $G(a)$ is a three-dimensional absolutely almost simple subgroup of G. In this situation we set, moreover, $T(a) = G(a) \cap T$, $T_a = \{t \in T \mid a(t) = 1\}$. We denote by W the Weyl group of $\Sigma(G, T)$ (when T is maximal) and by Δ a system of simple roots in Σ. Then w_0 denotes the only element of W which maps positive roots with respect to Δ into negative.

Finally, $|S|$ denotes the cardinality of S and $\mathbf{Z}/2$ denotes the group of two elements.

1. Preliminaries: Properties of the Field and of a Division Quaternion Algebra

1.1. Lemma. (i) $\operatorname{char} k = 0$

(ii) $[k^* : k^{*2}] = 2$, $k^* = k^{*2} \cup -k^{*2}$

(iii) $k^{*4} = k^{*2}$.

Proof. Suppose char $k = p \neq 0$. Then $-1 \in \mathbf{F}_p \subset k$ and $\mathbf{F}_p{}^2 - \mathbf{F}_p{}^2 = \mathbf{F}_p \ni -1$. So R2 does not hold in this case. This proves (i). By R1 we have $k^* = k^{*2} \cup -k^{*2}$ (because k^* has only one square nonresidue) whence (ii). If $a \in k^{*2}$ we have $a = b^2$, $b \in k^*$. We can assume that $b \in k^{*2}$ (because of (ii)). Then $b = c^2$, i.e. $a = c^4$, i.e. $k^{*2} = k^{*4}$, as required in (iii).

1.2. LEMMA. (i) $K^{*2} = K^*$

(ii) $N(K^*) = k^{*2}$

(iii) $K^{1^2} = K^1$

(iv) $K^* = k^{*2} \times K^1$.

Proof. Since K^* has no quadratic extensions we have $K^* = K^{*2}$. We have next that $N(K^*) \subset k^{*2}$ (trivially) and $N(K^*) \not\ni -1$ (because of R2). So (ii) follows from 1.1(ii). To prove (iii) take $h \in K^1$. By (i) there exists $m \in K^*$ with $h = m^2$. We have $N(m^2) = 1$ whence $N(m) = \pm 1$. Since $-1 \notin N(K)$ we have $m \in K^1$ whence $K^1 = K^{1^2}$. Consider k^{*2} and K^1 as subgroups of K^*. We have $k^{*2} \cap K^1 = 1$ since for $h \in k^{*2} \cap K^1$ we have $N(h) = h^2 = 1$, whence $h = \pm 1$. Now $-1 \notin k^{*2}$ implies that $k^{*2} \cdot K^1 = k^{*2} \times K^1$. On the other hand $N: K^* \to k^*$ has kernel K^1 and we have by 1.1(iii) that $N(K^*) = N(k^{*2} \cdot K^1)$. This proves (iv).

1.3. Let D be the quaternion algebra over k which is split by K and corresponds to norm residue $-1 \in k^*/N(K^*)$.

LEMMA. *Any quaternion division algebra over k is isomorphic to D.*

Proof. First, D is a division algebra since its norm form Nrd does not represent zero by R2. Second, if $\tilde{D}$ is another quaternion division algebra over k, then $\tilde{D}$ is split by K since K is the only quadratic extension of k and $\tilde{D}$ corresponds to some non-identity element of $k^*/N(K^*)$. But this latter group contains only one non-identity element, namely $-k^{*2}$. So there is only one division algebra, namely D.

1.4. LEMMA. (i) *all maximal subfields of D are isomorphic to K*

(ii) $D^{1^2} = D^1$

(iii) $NrdD^* = k^{*2}$

(iv) $D^* = k^{*2} \times D^1$.

Proof. Any maximal subfield of D is a quadratic extension of k. So (i) follows from R1. Since D is the union of its maximal subfields and since restriction of Nrd to a maximal subfield is the norm of that field, (ii) and (iii) follow from 1.2(ii), 1.2(iii). The above also implies that $k^{*2} \cdot D^1 = k^{*2} \times D^1$. Now as in the proof of 1.2(iv) we have $Nrd(k^{*2} \cdot D^1) = Nrd(D^*) = k^{*2}$, whence (iv).

2. Properites of Groups of Type A_1 over k

We consider an algebraic anisotropic k-group G of type A_1. If G is simply connected we denote it sometimes by $\tilde{G}$ and we have $G(k) = D^1$. If G is adjoint we denote it sometimes by $\bar{G}$ and we have $\bar{G}(k) = D^*/k^*$. We use here results of [18]. In notations of [18] we have $D = D_{-1}$.

2.1. Lemma. *Any two maximal k-tori of G are conjugate by an element of $G(k)$.*

Proof. It follows from [18, 1.3.5] for $\bar{G}$ and from 1.4(iii) together with [18], 5.2.3 for $\tilde{G}$.

2.2. Lemma. *If T is a k-subtorus of G then*

$$N_{G(k)}(T(k))/T(k) \simeq \mathbf{Z}/2\mathbf{Z}.$$

Proof. It follows from [18, 1.3.5] for $\bar{G}$ and from [18, 5.2.2] (together with the fact that $a = -1$ in our case) for $\tilde{G}$.

2.3. Lemma. *$G(k)$ acts travitively on the set of Borel K-subgroups of G.*

Proof. It follows from [18, 1.3.6, 5.2.4].

2.4. Lemma. $\bar{G}(k) \simeq G(k)/\{\pm 1\}$.

Proof. We have $\bar{G}(k) = D^*/k^*$, i.e., by 1.4(iv) $\bar{G}(k) = (D^1 \times k^{*2})/k^* = D^1/(D^1 \cap k^*) = D^1/\{\pm 1\} = G(k)/\{\pm 1\}$.

2.5. Lemma. (i) $[G(k), G(k)] = G(k)$.

(ii) *$G(k)$ has no subgroups of finite index.*

Proof. Because of 2.4 it is sufficient to prove the Lemma for $G = \tilde{G}$. Then $G(k) = D^1$. So to prove (i) take $h \in D^1$. Let T be a maximal k-torus containing h, $m \in N_{G(k)}(T(k)) - T(k)$. Then $[m, T(k)] = (T(k))^2 \simeq K^{1^2} = K^1$ (by 1.2(iii)). So $[m, T(k)] = T(k)$, i.e. $h \in [G(k), G(k)]$, whence (i).

To prove (ii) note that since $T(k) \simeq K^1$ is 2-divisible for any k-subtorus T of G (by 1.2(iii)), the order of any finite quotient of $G(k)$ is odd. Therefore by Feit–Thompson theorem any finite quotient is solvable. Now (i) implies that $G(k)$ has no finite quotients.

2.5.1. *Remark.* In view of E. Artin [1, Chap. V, Sect. 3] Lemma 2.5 may possibly be generated to: any simple quotient of $\tilde{G}(k)$ is siomorphic to $G'(k')$ where G' is a group of type A_1 defined over an archimedean field k'.

3. Properties of Admissible Algebraic k-Group

The considerations of this section are heavily based on the earlier paper [17]. In order to avoid repetitions we give only minimum of definitions. We refer to [17] for details.

Let G be a semi-simple k-group and let T be a maximal k-subtorus of G. We say that T (resp. G) is *admissible* if it is anisotropic over k and split over K. Any admissible group contains an admissible torus. If T is an admissible torus of G and $\Sigma = \Sigma(G, T)$ is the root system of G with respect to T, then $\overline{} \in \mathrm{Gal}(K/k)$ acts on Σ by $\bar{a} = -a$. If $\{x_a(t)\}_{a\in\Sigma}$ is a coherent parametrization of root subgroups of G (with respect to T), then we have $\overline{x_a(t)} = x_{-a}(d_a\bar{t})$ for $t \in K$ with $d_a \in k^*$. We write $\lambda_a = d_a N(K^*)$. The set $\{\lambda_a\}_{a\in\Sigma}$ determines G up to central k-isogeny (cf. [17] and Appendix 2, Theorem 8.8 and Corollary 8.9).

Since $\bar{a} = -a$ for $a \in \Sigma$, the three-dimensional subgroups $G(a) = \langle x_a(\bar{k}), x_{-a}(\bar{k})\rangle$ are defined over k. An admissible subtorus T' of G is said to be *associated with T (with respect to $a \in \Sigma$)* if it is contained in $G(a) \cdot T$. We fix a Borel K-subgroup containing T. Let $\Sigma^+ = \Sigma(B, T)$ and let Δ be the corresponding system of simple roots. We also fix a decomposition $w_0 = s_{a_1} \cdots s_{a_m}$, $a_i \in \Delta$, and we denote the parabolic $B \cdot G(a_i)$ by P_i.

3.1. Proposition. *For every reduced root system Σ there exists and up to isomorphism only one admissible simply connected group with root system Σ. This group is almost simple over k if and only if Σ is irreducible.*

Proof. To construct such group we have by [17. no. 6] and Appendix 2 below to ascribe to each $a \in \Sigma$ an element $d_a \in k^*$. These numbers must satisfy relations ([17, no. 14] and Appendix 2 below), namely $d_{a+b} = -d_a d_b$ if $a, b, a+b \in \Sigma$ and $d_{-a} = d_a^{-1}$. And any system of numbers satisfying these relations determines a k-form G of a split group with root system Σ. Moreover, G contains an admissible torus. Let us set $d_a = -1$ for all $a \in \Sigma$. Then the relations are satisfied and we obtain a group G. Let us prove that G is admissible (i.e. anisotropic). Note first that by [17, no. 8] and 1.4(iii), the set $\{\lambda'_a\}_{a\in\Sigma(G,T')}$ corresponding to an admissible torus T' associated with T is the same: $\lambda'_a = -N(K^*)$. If G were isotropic then by ([17], nos. 9, 7) there would exist an admissible torus $\tilde{T}$ which could be connected with T by a finite sequence of associated tori, and a root $b \in \Sigma(G, \tilde{T})$ such that $\tilde{\lambda}_b \in N(K^*)$. Since on each step of associations we have $\lambda'_a = -N(K^*)$ the above is impossible and so G is anisotropic. Since $\overline{}$ acts as -1 on Σ, G is almost simple over k if and only if Σ is connected.

It remains to show the unicity. It is easy since the isomorphism class of G is completely determined by the set $\{\lambda_a = d_a \cdot N(K^*)\}_{a\in\Sigma}$. And by 1.2(ii) we can choose $\{\lambda_a\}$ in only one way so that $\lambda_a \notin N(K^*)$.

3.2. LEMMA. *Let G be an admissible group and T an admissible subrotus of G. Then*

$$N_{G(k)}(T(k))/T(k) = N_G(T)/T.$$

Proof. It follows from 2.2 since $N_G(T)$ is generated by the groups $N_{T\cdot G(a)}(T)$, $a \in \Sigma$.

3.3. LEMMA. *Let G be an admissible group. Let* $w_0 = s_{a_1} \cdots s_{a_m}$, $a_i \in \Delta$, *and* $P_i = G(a_i)B$. *Then* $G(K) = P_1(K) \cdot P_2(K) \cdots P_m(K)$.

Proof. The right hand side is invariant under right and left multiplications by $B(K)$. Since $G(K) = B(K) \cdot W \cdot B(K)$ it is sufficient to establish that every $w \in W$ is a product of reflections in some a_i taken in the same order as the given expression for w_0. It is known (and follows by descending induction from [4, Chap. 6, Sect. 1, Proposition 17] and exchange property [4, Chap. 4, Sect. 1, no. 1.5]).

3.4. PROPOSITION. *Let G be an admissible group. Let B and* a_i *be as in* 3.3 *and let B′ be a Borel K-subgroup of G. Then there exists* $g \in G(a_1)(k) \cdots G(a_m)(k)$ *such that* $gBg^{-1} = B'$. *In particular,* $G(k)$ *acts transitively on* $(G/B)(K)$.

Proof. (Compare [17, nos. 7, 11]). Apply Lemma 3.3 to the decomposition $w_0(=w_0^{-1}) = s_{a_m} \cdots s_{a_1}$. Take $p_i \in P_i(K)$ such that $g = p_m p_{m-1} \cdots p_1$. Set $d_i = p_i \cdots p_1$, $B_0 = B$, $B_i = d_i B d_i^{-1}$, $T_i = B_i \cap \bar{B}_i$, $R_i = d_i P_i d_i^{-1}$, $G_i = R_i \cap \bar{R}_i$. Then T_i is defined over k and split over K. Since G is admissible it follows that so is T_i. Then G_i is a three-dimensional k-subgroup normalized by T_i. By 2.3 there exists $h_i \in G_i(k)$ such that $B_i = h_i B_{i-1} h_i^{-1}$ for $i \geqslant 1$. Therefore $B_i = (h_i \cdots h_1)\, B (h_i \cdots h_1)^{-1}$, $R_i = (h_i \cdots h_1)\, P_i (h_i \cdots 1 h_1)^{-1}$. Thus

$$G_i = (h_i \cdots h_1)\, G(a_i)(h_i \cdots h_1)^{-1}.$$

Hence $h_{i+1} = (h_i \cdots h_1)\, g_i (h_i \cdots h_1)^{-1}$ for some $g_i \in G(a_i)(k)$. We have $g_1 = h_1$, $h_2 h_1 = h_1 g_2 h_1^{-1} \cdot h_1 g_2 = g_1 g_2$. Suppose that we have established that $h_i \cdots h_1 = g_1 \cdots g_i$. Then $h_{i+1} \cdots h_1 = h_{i+1} g_1 \cdots g_i = (g_1 \cdots g_i)\, g_{i+1} (g_1 \cdots g_i)^{-1} \cdot (g_1 \cdots g_i) = g_1 \cdots g_{i+1}$. Therefore by induction we have $h_m \cdots h_1 = g_1 \cdots g_m$. Since $g_i \in G(a_i)(k)$, our assertion follows.

3.5. COROLLARY. *Let G be admissible. All pairs* (T, B) *consisting of an admissible torus T and a Borel K-group, containing it, are conjugate by elements of* $G(a_1)(k) \cdots G(a_m)$. *In particular, all admissible tori are conjugate by* $G(k)$.

Proof. Let (T, B), (T', B') be two pairs of the described type. Then there exists $h \in G(a_1)(k) \cdots G(a_m)(k)$ such that $hBh^{-1} = B'$. Since $T = B \cap \bar{B}$, $T' = B' \cap \bar{B}'$, $h \in G(k)$ our assertion follows from the preceding one.

3.6. COROLLARY. *For an admissible G one has*

$$G(k) = T(k) \cdot G(a_1)(k) \cdots G(a_m)(k).$$

Proof. Let $g \in G(k)$. Set $T = gTg^{-1}$, $B = gBg^{-1}$. Then by 3.5 there exists $h \in G(a_1)(k) \cdots G(a_m)(k)$ such that $hT'h^{-1} = T$, $hB'h^{-1} = B$, which means that $(hg)\,T(hg)^{-1} = T$, $(hg)\,B(hg)^{-1} = B$, i.e. $hg \in T(k)$. Since $T(k)$ normalizes all $G(a_i)(k)$ our assertion follows.

3.7. THEOREM. *Let G be simply connected and admissible, and let a_i be as in* 3.3. *Then $G(k) = G(a_1)(k) \cdots G(a_m)(k)$. In particular*

(i) $G(a)(k)$, $a \in \Delta$, *generate* $G(k)$

(ii) $[G(k), G(k)] = G(k)$.

Proof. Let us first show that (i) and (ii) follow from the main assertion. This is clear for (i). To prove (ii) we remark first that

$$G(k) \supset [G(a_1)(k), G(a_1)(k)] \cdots [G(a_m)(k), G(a_m(k)]$$

and the last expression coincides with $G(a_1)(k) \cdots G(a_m)(k)$ because of 2.5.

The rest of the theorem follows from 3.6 and the following

3.7.1. LEMMA. *Set $T(a) = G(a) \cap T$. Then (for simply connected admissible G) we have: $T(k)$ is the direct product of groups $T(a)(k)$, $a \in \Delta$.*

Proof. By [14, Sect. 3, Lemma 28c] we know that $T(K)$ is the direct product of groups $T(a)(K)$, $a \in \Delta$. Since the action of $\mathrm{Gal}(K/k)$ preserves all $T(a)$, we have our assertion.

3.7.2. *Remark.* V. Kac proved in [7] for compact Lie groups (i.e., in the case $k = \mathbf{R}$) that modulo a finite central group all relations between groups $G(a_i)(\mathbf{R})$ (when they generate $G(\mathbf{R})$) follow from relations in groups of $\mathbf{C}$-rank 2. He thinks that the ideas of his proof may work also in our slightly more general case.

4. HOMOMORPHISMS OF GROUPS OF TYPE A_1

Let k be a field satisfying R1, R2 and let G be an absolutely almost simple algebraic k-group of type A_1. Let k' be another infinite field and G' an algebraic absolutely almost simple k-group. Let $\varphi\colon G(k) \to G'(k')$ be a homomorphism with dense image.

4.1. THEOREM. *There exist*

(i) *A unique place* $\varphi: k \to k'$; *let A be its valuation ring*;

(ii) *A unique structure* G_A *of a semi-simple group scheme over A on G*; *we have* $G_A(A) = G(k)$ *in this structure*;

(iii) *A unique structure of an algebraic* $\varphi(A)$*-group on* G';

(iv) *A unique central* $\varphi(A)$*-isogeny* β *of algebraic* $\varphi(A)$*-groups,* $\beta: {}^{\varphi}G_A = G_A \otimes_{\varphi} \varphi(A) \to G'_{\varphi(A)}$ *such that*

$$\alpha(g) = \beta(\varphi^0(g)) \quad \text{for} \quad g \in G(k).$$

We shall obtain more information in 4.5. The present formulation can be derived in a somewhat more general form, although we can not globalize (to groups of other type) those more general results. We have chosen to prove them here because the proofs exhibit the assumptions we are actually using. Our argument is based on results of [18] about projective plane structure on the set of minimal centralizers in adjoint groups of type A_1. We start with

4.2. Lemma. *In conditions of* 4.1 *assume that G is adjoint. Then G′ is of type* A_1. *Moreover if* char $k' \neq 2$ *then G′ is also adjoint.*

Proof. This is [18, 4.1(i), 4.2.2].

4.3. Now let k and k' be infinite fields. Let G and G' be respectively k- and k'-forms of $PGL(2)$. Let $\alpha: G(k) \to G'(k')$ be a homomorphism with dense image. Assume that α satisfies the following condition:

H1. The kernel of α does not contain $H(k)$ for any connected one-dimensional k-subgroup of G.

This assumption on α obviously follows from a stronger assumption on $G(k)$:

H1bis. No normal subgroup of $G(k)$ contains any $H(k)$, H one-dimensional connected k-subgroup of G.

Our aim here is to prove

4.3.1. Theorem. *Suppose that G is anisotropic and* α *satisfies* H1. *Then conclusions of* 4.1 *hold for* α, $G(k)$, $G'(k')$.

4.3.2. *Remark.* There is no actual loss of generality in assumption that G is anisotropic. Because in the isotropic case $[G(k), G(k)]$ is simple, so the kernel can not be big. And it follows from [2] that $A = k$ and φ is a homomorphism in this case.

4.3.3. Let G be a k-form of $PGL(2)$. Let us recall some notations and results of [18]. These and some other results of [18] are used without reference in 4.3.3–4.3.12 below. A subgroup $M = Z_{G(k)}(h)$, $h \in G(k)$, is called a *minimal centralizer* if $M = \bigcup_{m \in M} Z_{G(k)}(m)$. Every element of $G(k)$ is contained in a unique minimal centralizer. The set of minimal centralizers is denoted $S(G(k))$. A subset L of $S(G(k))$ is an *involutorial line* if $|L| > 2$ and there exists $h \in G(k)$, $h^2 = 1$ such that $hmh^{-1} = m^{-1}$ for all $m \in M$ for any $M \in L$; in this case we write $L = L(h)$. A subset L of $S(G(k))$ is called a *parabolic line* if $|L| > 2$ and L consists of all minimal centralizers contained in a normalizer of some minimal centralizer. A subset L is called a *line* if it is either a parabolic or an involutorial line. It is proved in [18] that $S(G(k))$ with lines defined as above is the projective plane $\mathbf{P}^2(k)$. All lines are involutorial if G is anisotropic. In characteristic 2 there is a special line, called *unipotent*. It is involutorial line corresponding to involution 1. Elements of all minimal centralizers of this line are unipotent and of order 2.

4.3.4. Let us begin our study with the following trivial observation:

Remark. Let M be a subgroup of $G(k)$, $x \in G(k)$. If x inverts elements of M then $\alpha(x)$ inverts elements of $\alpha(M)$. In particular either $\alpha(x)$ centralizes $\alpha(M)$ and then $\alpha(M)$ is of period 2, or $\alpha(M)$ is not of period 2 and then $\alpha(x)$ does not centralize $\alpha(M)$.

4.3.5. LEMMA. *Let T be a k-torus of G. Then $\alpha(T(k))$ is infinite.*

Proof. Denote $R = N_{G(k)}(T(k)) - T(k)$. For every involutotial line L of $S(G(k))$ passing through $T(k)$ there exists $x \in R$ such that x acts as inversion on every $M \in L$. Therefore either $\alpha(M) \in L(\alpha(x))$ or $\alpha(M) \subseteq Z_{G'(k')}(\alpha(x))$ and $\alpha(M)$ is of period 2 in this latter case. Let $\underset{\sim}{Q}'$ be the set of elements of order 2 in G'. Since all lines are involutorial and since lines through $T(k)$ cover all of $G(k)$ the above argument says that $\alpha(G(k)) \subset \bigcup_{x\in R} L(\alpha(x)) \cup \underset{\sim}{Q}'$. Now $T(k)$ acts simply transitively on R by multiplications on the right. Therefore if $|\alpha(T(k))| < \infty$, then $\#\{\alpha(x) | x \in R\} < \infty$ whence it follows that $\alpha(G(k))$ is contained in a union of a finite number of proper closed subvarieties of G', i.e. $\alpha(G(k))$ is not dense, a contradiction.

4.3.6. COROLLARY. *Let T be a k-torus of G. If* char $k' \neq 2$ *then $\alpha(T(k))$ is not of period* 2.

Proof. The group G', being a form of $PGL(2)$, contains infinite subgroups of period 2 only if char $k' = 2$.

4.3.7. LEMMA. *If U is a unipotent k-subgroup of G (in particular, if* char $k = 2$*) then $\alpha(U(k))$ is infinite.*

Proof (the same as in 4.3.5). Every line L of $S(G(k))$ through $U(k)$ is involutorial and has the form $L = L(u)$, $u \in U(k)$. Let R' be the set of nonidentity elements of $\alpha(U(k))$ and let $\underset{\sim}{Q}'$ be the set of elements of order 2 in G'. Then $\alpha(G(k)) \subset \bigcup_{h'\in R'} L(h') \cup \underset{\sim}{Q}'$. So if R' is finite then $\alpha(G(k))$ is not dense.

4.3.8. Corollary. *If* char $k = 2$ *then* char $k' = 2$.

Proof. Let U be a unipotent k-subgroup of G. Then $\alpha(U(k))$ is an infinite subgroup of G' of period 2. It is possible only if char $k' = 2$.

4.3.9. Corollary. *For every minimal centralizer M in $G(k)$ there exists a unique minimal centralizer in $G'(k')$ containing $\alpha(M)$.*

Proof. If $\alpha(M)$ is not of period 2 take $h \in \alpha(M)$ such that $h^2 \neq 1$. Then $M' = Z_{G'(k')}(h)$ is a minimal centralizer and $M' \supset \alpha(M)$. If char $k' \neq 2$ then char $k \neq 2$ (by 4.3.8) and by 4.3.5 every $\alpha(M)$ is not of period 2. So assume that char $k' = 2$ and $\alpha(M)$ is of period 2. Then $M' = Z_{G'(k')}(h)$ is a minimal centralizer for every $h \in \alpha(M)$, $h \neq 1$, and $\alpha(M) \subseteq M'$.

4.3.10. Proposition. *For a minimal centralizer M of $G(k)$ let $\bar{\alpha}(M)$ be the unique minimal centralizer of $G'(k')$ containing $\alpha(M)$. Then $\bar{\alpha}$: $S(G(k)) \to S(G'(k'))$ is a homomorphism of projective spaces* (*cf. Appendix* 1).

Proof. We have to show that the image of a line is contained in a line. Since G is anisotropic all lines are involutotial. Let $x \in G(k)$, $x^2 = 1$, $x \neq 1$, and let $L(x)$ be the corresponding involutorial line. If $\alpha(x) = 1$ then $\alpha(M)$ is of period 2 for all $M \in L(x)$. Therefore char $k' = 2$ and $\alpha(L(x))$ is contained in the unipotent line $L(1)$ of $S(G'(k'))$. Next assume that $\alpha(x) \neq 1$. Then $L' = L(\alpha(x))$ contains all $\alpha(M)$, $M \in L(x)$, such that $\alpha(M)$ is not of period 2 or $\alpha(M) \subset Z_{G'(k')}(\alpha(x))$ (when char $k' = 2$), when $\alpha(M) \subset L(\alpha(x))$ for all $M \in L(x)$. It remains to consider the case when char $k = 2$ and the line in question is the unipotent line. In this case our assertion follows from 4.3.7, 4.3.8.

4.3.11. The above proposition enables us to apply the results of Appendix 1. Let φ, A, I, M, V, V', $\bar{\beta}$, $\tilde{\beta}$ be the same as there. We lift G to the group $SO(F, V)$ of special orthogonal transformations of V (recall, that $SO(F, V)$ is mapped isomorphically to G) with respect to a non-degenerate quadratic form F on V. We choose F (as we may by multiplying by elements of k) so that F would be integral with respect to M and so that $F \not\equiv 0 \bmod I$.

Lemma. *$SO(F, M)$ is a semi-simple group scheme over A such that* (i) $SO(F, M)(A) = SO(F, V)(k)$, (ii) *$\alpha$ is the composition of φ^0 and of the isomorphism of ${}^{\varphi}(\mathbf{P}GL(3)_A)$ to $\mathbf{P}GL(3)_k$ induced by $\tilde{\beta}$.*

Proof. Let $g \in SO(F, V)(k)$. Since the action of G commutes with the map $\bar{\alpha}$, g must map M into some $a(g) \cdot M$, $a(g) \in k$ (according to 7.3.7). Since $SO(F, V)$ is unimodular we have $a(g) = 1$ for all $g \in SO(F, V)(k)$, whence $SO(F, V)(k)$ preserves M. Evidently, it also preserves F.

Consider now the situation over k'. The map α agrees with $\bar{\alpha}$ in the sense that for $g \in G(k)$ and $x \in \mathbf{P}^2(k) = S(G(k))$ we have $\alpha(g)\, \bar{\alpha}(x) = \bar{\alpha}(gx)$. We know the structure of $\bar{\alpha}$ from Appendix 1, $\bar{\alpha} = \tilde{\beta} \circ \varphi^0$. It follows that the image of $SO(F, V)(k)$ is obtained by applying φ^0 and then group isomorphism of two $\mathbf{P}GL(3)_{k'}$. Now the image of $SO(F, V)(k)$ under φ^0 preserves $\varphi^0(F)$. On the other hand G' is irreducible if char $k' \neq 2$ and has only one fixed point in $\mathbf{P}^2_{k'}$ if char $k' = 2$. Since the image of $G(k)$ in G' is dense the same holds for $\alpha(G(k))$. If char $k' \neq 2$ and the form $\varphi^0(F)$ were degenerate then $\varphi^0(G(k))$ would fix some subspace of $\mathbf{P}^2(k')$, which is impossible. So if char $k' \neq 2$ then $\varphi^0(F)$ is non-degenerate. Suppose that char $k' = 2$. Since all derived groups of $G'(k')$ are non-trivial and since $\alpha(G(k))$ is dense in G' the same holds for $\alpha(G(k))$, whence $\varphi^0(G(k))$ can not be solvable. Since $F \not\equiv 0 \bmod I$ i.e., $\varphi^0(F) \neq 0$, we can only have that defect of $\varphi^0(F)$ is 1.

Since $\varphi^0(F)$ is non-degenerate if char $k' \neq 2$ and $\varphi^0(F)$ is of defect 1 if char $k' = 2$, the group scheme $SO(F, M)$ is indeed semi-simple (its fibers are smooth and the special fiber is semi-simple by the above argument; hence all fibers are semi-simple since the property of being semi-simple is open, cf. [5], X. 8, p. 121) or ([5], XIX, Cor. 2.6). This proves our lemma.

4.3.12. Let us conclude the proof of Theorem 4.3.1. Actually, what is left, namely parts (iii) and (iv) of Theorem 4.1 are easy corollaries of Theorem 7.4 of Appendix 1. Since it follows from 7.4 that $\alpha(G(k))$ is contained in $\mathbf{P}GL(3, \varphi(A))$, its closure is defined over $\varphi(A)$, i.e. G' is defined over $\varphi(A)$. Now $\tilde{\beta}$ is defined over $\varphi(A)$ and the isomorphism $\beta: {}^{\varphi}\mathbf{P}GL(3)_A \to \mathbf{P}GL(3)_{\varphi(A)}$ induced by $\tilde{\beta}$ is also defined over $\varphi(A)$ whence our last assertion.

4.4. Let us now consider the case when G is simply connected. We impose additional (extremely strong, cf. 4.4.4 below) condition on k and G.

Let k be an infinite field, G an anisotropic k-form of $SL(2)$, $\bar{G}$ the adjoint group of G. Our condition is

H2. The natural map $G(k) \to \bar{G}(k)$ is surjective.

4.4.1. THEOREM. *Let k' be another infinite field, let G' be an absolutely almost simple k'-group of type A_1. Let $\alpha: G(k) \to G'(k')$ be a homomorphism with dense image. Suppose that α satisfies* H1 *and G satisfies* H2. *Then the conclusions* (i) *through* (iv) *of Theorem* 4.1 *hold and in addition we have*

(v) *there exists a unique map $\mu: G(k) \to C(G'(k'))$ such that*

$$\alpha(g) = \mu(g) \cdot \beta(\varphi^0(g)) \qquad \textit{for} \quad g \in G(k).$$

As before proof will be given in several steps.

4.4.2. LEMMA. *Theorem* 4.4.1 *holds if G' is adjoint.*

Proof. Since G' is adjoint we have $C(G'(k')) = 1$, whence $C(G(k)) = \{\pm 1\} \subset \operatorname{Ker} \alpha$. Therefore α can be factored through the map $G(k) \to \bar{G}(k)$. By

H2 this map is onto and therefore 4.3.1 is applicable. It provides $\bar{G}(k)$ with the structure of a group scheme over A. This structure lifts to G (use Appendix 2, for example). Clearly the canonical map $G \to \bar{G}$ commutes with φ^0. Therefore we get that α can be factored as φ^0 times a central $\varphi(A)$-isogeny as required. Note that μ has to be trivial in this case as $C(G') = 1$.

4.4.3. LEMMA. *Theorem* 4.4.1 *holds if* G' *is a form of* $SL(2)$.

Proof. Let $\bar{G}'$ be the adjoint group of G and let $\kappa' \colon G' \to \bar{G}'$ be the canonical projection. Then 4.4.2 applies to $\alpha' = \kappa' \circ \alpha$. Let β' be the central $\varphi(A)$-isogeny $\beta' \colon {}^{\varphi}G_A \to \bar{G}'$, guaranteed by 4.4.2. Then we can construct (using, for example, 8.7 of Appendix 2) the $\varphi(A)$-isomorphism $\beta \colon {}^{\varphi}G_A \to G'$ such that $\beta' = \kappa' \circ \beta$. Let us now compare $\beta \circ \varphi^0$ and α. Since $\beta' \circ \varphi^0$ and $\kappa' \circ \alpha$ coincide we get $[(\beta \circ \varphi^0)(g)]^{-1} \cdot \alpha(g) \in C(G'(k'))$. Denote $\mu(g) = [(\beta \circ \varphi^0)(g)]^{-1} \cdot \alpha(g)$. Then $\mu \colon G(k) \to C(G'(k'))$ and we are done.

4.4.4. Now let us show that the condition H2 is very strong. We use notations of [18].

LEMMA. *Let* H2 *be satisfied for* G, *a form of* $SL(2)$, *and let* D *be the corresponding quaternion division algebra over* k. *Then*

(i) D *is split by* $K = k((-1)^{1/2})$ (*in particular*, -1 *is not a square in* k);

(ii) D *is determined by* $-1 \in k^*/N_{K/k}(K^*)$ (*i.e.*, $D = D_{-1}$)

Proof. Let $\tilde{K}$ be any quadratic separable splitting field of D and let D be determined by $a \in k^*/N_{\tilde{K}/k}(\tilde{K}^*)$. Let T be the maximal torus of G corresponding to $\tilde{K}$. Since $\kappa \colon G(k) \to \bar{G}(k)$ is surjective and since $N_{\bar{G}(k)}(\kappa(T)) \neq (\kappa(T))(k)$, it follows that $N_{G(k)}(T(k)) \neq T(k)$. Therefore $-a \in N_{\tilde{K}/k}(\tilde{K}^*)$ (cf. [18], 5.2.2). Thus we can replace a by $-1 = a \cdot (-a)^{-1} \in a \cdot N_{\tilde{K}/k}(\tilde{K}^*)$, i.e., we can assume that $a = -1$. Then $K = k((-1)^{1/2})$ splits D and applying the above argument to $\tilde{K} = K$ we obtain that $D = D_{-1}$.

4.4.5. COROLLARY. *Zero is not a sum of four squares in* k.

Proof. The norm form of D does not represent zero since D is division and it is a sum of four squares.

4.4.6. *Remark.* We can restate 4.4.4 saying that D has to be Hamiltonian (i.e., split by $K = k((-1)^{1/2})$ and determined by $-1 \in k^*/N_{K/k}(K^*)$).

4.5. Let us now apply the above discussion to the case at hand, i.e., to the fields satisfying R1, R2.

4.5.1. LEMMA. *The conditions* H1bis *and* H2 *are satisfied.*

Proof. Suppose that M is a proper normal subgroup of $G(k)$, containing $H(k)$ where H is a one-dimensional connected k-subgroup. Then by 2.1, M contains all of $G(k)$, a contradiction. Now H2 follows from 2.4.

4.5.2. PROPOSITION. *Suppose that G' is adjoint. Then G' is an anisotropic $\varphi(A)$-group and the corresponding quaternion division algebra D' is Hamiltonian.*

Proof. Let $\bar{\alpha}$: $S(G(k)) \to S(G'(\varphi(A)))$ be the homomorphism of projective planes induced by α. Then the image of $S(G(k))$ under $\bar{\alpha}$ is a projective subplane of $S(G'(\varphi(A)))$. So the image is $\mathbf{P}^2(k'')$. But we know that $\varphi(A) = k''$. So $\bar{\alpha}(S(G(k))) = S(G'(\varphi(A)))$. But $G(k)$ acts transitively on $S(G(k))$ (by 2.1). Therefore $G'(\varphi(A))$ acts transitively on $S(G'(\varphi(A)))$. Therefore all connected one-dimensional $\varphi(A)$-subgroups of G' are tori, whence G' is anisotropic over $\varphi(A)$. Moreover all $\varphi(A)$-tori of G' are conjugate in $G'(\varphi(A))$, for every $\varphi(A)$-subtorus T' of G' we have $N_{G'(\varphi(A))}(T'(\varphi(A))) \neq T'(\varphi(A))$, and

$$[G'(\varphi(A)), G'(\varphi(A))] = G'(\varphi(A)).$$

It follows from the last equality that two former properties hold also for simply connected cover of G'. Now 4.4.4 implies the last statement of 4.5.2.

4.5.3. COROLLARY. char $k' = 0$.

4.5.4. LEMMA. *Theorem* 4.1 *holds.*

Proof. First we remark that $\mu = 1$ in 4.4.1 because of 2.5(i). Second we have to settle the case when G is adjoint and G' is simply connected, but this case cannot happen because of 4.2 and 4.5.3.

4.6. Let us now obtain a description of the structure of the group scheme over A on G in terms described in Appendix 2.

PROPOSITION. *The semi-simple group scheme G_A is given by data of Corollary* 8.9 *where $A_1 = A((-1)^{1/2})$.*

Proof. Let T be an arbitrary A-subtorus of G_A (all of them are conjugate). Then T is split by $A((-1)^{1/2})$ which is clearly unramified and quadratic. Let $M = M_1 \oplus M_2$ be the decomposition of the lattice M under action of T with $TM_i = M_i$, $T \mid M_2 = Id$. Then our quadratic form F (cf. 4.3.11) takes the form $F = x_1^2 - x_2^2 - ax_3^2$ (with $a \in A^\times$ since F is non-degenerate modulo I). Consider the maximal subtorus T_1 of G_A, corresponding to the splitting $M = Ae_1 \oplus (Ae_2 - Ae_3)$. Since T_1 is also split by $A((-1)^{1/2})$ we see that the restriction of F to $Ae_2 + Ae_3$ is of the form $b(x_2^2 + x_3^2)$ with $b \in A^*$. Therefore $a = 1$, i.e., $F = x_1^2 + x_2^2 + x_3^2$. This means that $d_{a,\sigma} = -1$ (in notations of Theorem 8.8 of Appendix 2).

5. Homomorphisms of Admissible Groups: Coincidence of Types

5.1. Theorem. *Let k be a field satisfying* R1, R2 *and let G be an admissible absolutely almost simple simply connected group over k. Let k' be an infinite field and let G' be an absolutely almost simple k'-group. Let $\alpha: G(k) \to G'(k')$ be a homomorphism with dense image. Then*

(i) $\operatorname{char} k' = 0$,

(ii) *for an admissible torus T of G the closure T' of $\alpha(T(k))$ in G' is a maximal subtorus of G',*

(iii) *there exists an isomorphism $\tilde{\alpha}: \Sigma(G, T) \to \Sigma(G', T')$ such that for any $d \in \Sigma(G, T)$ the closure of $\alpha(G(d)(k))$ is $G'(\tilde{\alpha}(d))$.*

The proof will be given in several steps. In this proof the main difficulty we want to overcome is to prove (ii). Note that if k were real closed then all maximal k-tori of G would be conjugate over k and 5.1(ii) would follow from the density of the $\alpha(G(k))$ in G and from the openness of the set of regular elements in G'.

Let us introduce some notation. Let T be an admissible torus in G, $\Sigma = \Sigma(G, T)$. For $a \in \Sigma$ we set $T(a) = T \cap G(a)$, $T_a = \{t \in T \mid a(t) = 1\}$. Let M', M'_a, $M'(a)$ be the closures of $\alpha(T(k))$, $\alpha(T_a(k))$, $\alpha(T(a)(k))$ in G'. Since $T(k)$ is commutative, M' also is commutative. Therefore M' contains a unique maximal torus, say T'. Then $T'_a = M'_a \cap T'$, $T'(a) = M'(a) \cap T'$ are unique maximal subtori in M'_a and $M'(a)$ respectively. Let $G'(a)$ be the closure of $\alpha(G(k))$ in G' and set $\Sigma' = \Sigma(G', T')$, $\Sigma'(a) = \Sigma(Z_{G'}(T'_a), T')$, $Z'(a) = Z_{G'}(T') \cap G'(\Sigma'(a))$.

5.2. Lemma. (i) *$G'(a)$ is connected and has no commutative quotients,*

(ii) *$G'(\Sigma'(a))$ is normalized by $Z_{G'}(T')$. In particular, $G'(\Sigma'(a)) \cdot Z_{G'}(T')$ is a group,*

(iii) *$G'(\Sigma'(a)) \cdot Z_{G'}(T')$ contains $G'(a)$,*

(iv) *if $a, b \in \Sigma$ are of the same length then $G'(a)$ and $G'(b)$ (resp., $G'(\Sigma'(a))$ and $G'(\Sigma'(b))$) are conjugate in G'.*

Proof. Suppose that $G'(a)$ is not connected. Then $G'(a)^0$ is of finite index in $G'(a)$. Since $\alpha(G(a)(k))$ is dense in $G'(a)$ we have

$$G'(a)/G'(a)^0 \simeq G(a)(k)/\alpha^{-1}(\alpha(G(a)(k)) \cap G'(a)^0).$$

But $G(a)(k)$ has no finite quotients by 2.5(ii). Therefore $G'(a) = G'(a)^0$. The fact that $G'(a)$ has no commutative quotients follows from 2.5(i) in the same way as above.

Clearly $G'(\Sigma'(a))$ is normalized by T' and by definition $G'(\Sigma'(a))$ is generated by non-trivial root subgroups) we see that $G'(\Sigma'(a))$ is a product of some semi-simple components of the semi-simple groups $[Z_{G'}(T'_a), Z_{G'}(T'_a)]$. Our statement (ii) follows from the fact $[Z_{G'}(T'_a), Z_{G'}(T'_a)] \supseteq [Z_{G'}(T'), Z_{G'}(T')]$, so that any semi-simple component of $Z_{G'}(T')$ is contained in some semi-simple component of $Z_{G'}(T'_a)$.

Now (iii) is clear because $G(a) = Z_G(T_a)$ and therefore $G'(a) \subseteq Z_{G'}(M'_a) \subseteq Z_{G'}(T'_a)$. Since $G'(a)$ does not have commutative quotients it follows $G'(a)$ is contained in some semi-simple components of $Z_{G'}(T'_a)$. All such components are contained in $G'(\Sigma'(a)) \cdot Z_{G'}(T')$.

Finally, (iv) follows from 3.2 and from transitivity of the Weyl group of Σ on the set of roots of the same length.

5.3. Lemma. (i) *$G'(a)/R_u(G'(a))$ is semi-simple of type $A_1 \times A_1 \times \cdots \times A_1$ ($r(a)$ times),*

(ii) *char $k' = 0$ (in particular, 5.1(i) holds).*

Proof. Let H' be an (absolutely almost) simple component of $G'(a)/R_u(G'(a))$. Then the composition of α with the natural projection maps $G(a)(k)$ to H' with dense image. We apply to this homomorphism Theorem 4.1 and Corollary 4.5.3 and obtain our statement. Note that the fact that $G'(a)/R_u(G'(a))$ is semi-simple follows from 5.2(i).

5.3.1. *Remark.* Note that 5.3(i) follows essentially from [18], 4.1(i) (because of the condition H2). Since 5.3 above is the only reference to the Section 4 in this Section, we use here essentially only part (i) of Theorem 4.1. The precise statement of 4.1 will be used in the next Section.

5.4. Lemma. *The numbers $r(a)$ do not depend on $a \in \Sigma$. Their common value is denoted r.*

Proof. By 5.2(iv) $r(a) = r(b)$ if a and b have the same length. On the other hand since $T(k)$ is the direct product of $T(a)(k)$, $a \in \Delta$, (by 3.7.1) we have $\dim T' = \Sigma_{a\in\Delta} r(a)$. Let $\tilde{\Sigma}$ be the subsystem of long roots in Σ and let $\tilde{\Delta}$ be its system of simple roots. Then $G(\tilde{\Sigma})$ is also simply connected and therefore $T(k)$ is the direct product of $T(d)(k)$, $d \in \tilde{\Delta}$. Thus $\dim T' = \Sigma_{d\in\tilde{\Delta}} r(d)$. Since $r(b)$ depends only on length of b, it follows from $\Sigma_{a\in\Delta} r(a) = \Sigma_{d\in\tilde{\Delta}} r(d)$ that $r(a) = r(d)$ for any pair $a, d \in \Sigma$, as asserted.

5.5. Lemma. *$\Sigma'(a) \cap \Sigma'(b) = \varnothing$ for $a, b \in \Sigma$, $a \neq \pm b$.*

Proof. Suppose $R = \Sigma'(a) \cap \Sigma'(b) \neq \varnothing$ for some $a, b \in \Sigma$, $a \neq \pm b$. Then T'_a, T'_b are contained in $T'_R = \{t \in T' \mid r(t) = 0 \text{ for all } r \in R\}$. Since $R \neq 0$ we have $T'_R \neq T'$. Therefore $T'_a \cdot T'_b \neq T'$. But $T_a \cdot T_b = T$ whence $T(k)/T_a(k) \cdot T_b(k)$ is periodic of bounded period. Therefore $\alpha(T(k))/\alpha(T_a(k)) \cdot \alpha(T_b(k))$ is periodic. which is impossible if $\dim T'/T'_a \cdot T'_b \geqslant 1$. This proves our claim.

5.6. PROPOSITION. (i) $\Sigma' = \bigcup_{a\in\Sigma^+} \Sigma'(a)$

(ii) $R_u(G'(d)) = 1$

(iii) *T' is a maximal torus of G'.*

Proof. Choose and fix in G' two opposite parabolic subgroups P^+ and P^- whose Levi component is $Z_{G'}(T')$. Let $U^\pm$ be their unipotent radicals. For every $d \in \Sigma$, set $U^\pm(d) = U^\pm \cap G'(\Sigma'(d))$, $P^\pm(d) = P^\pm \cap G'(\Sigma'(d))$. Since T' normalizes $G'(\Sigma'(d))$ we have that $P^\pm(d)$ are parabolic subgroups of $G'(\Sigma'(d))$ and $U^\pm(d) = R_u(P^\pm(d))$. Set $S(d) = G'(\Sigma'(d))/P^+(d)$. We have $S(d) = G'(\Sigma'(d)) \cdot Z_{G'}(T')/P^+(d) \cdot Z_{G'}(T')$ (recall that $Z_{G'}(T')$ normalizes both $G'(\Sigma'(d))$ and $P^+(d)$). The canonical map $p(d)$: $G'(\Sigma'(d)) \cdot Z_{G'}(T') \to S(d)$ is a locally trivial fibration. Let $\tilde{U}^-(d)$ be a local section of this fibration over $p(d)(G'(d))$. The set $V(d) = Z_{G'}(T') \cdot U^-(d)\, U^-(d)$ contains an open subset of $G'(\Sigma'(d))$ and the set $\tilde{V}(d) = Z_{G'}(T')\, U^+(d) \cdot \tilde{U}^-(d)$ contains an open subset of $G'(d)$. Set now $\tilde{q}(d) = \dim \tilde{U}^+(d)$, $q(d) = \dim U^+(d)$, $q = \dim Z_{G'}(T')$. Because of 5.2(iv) the numbers $\tilde{q}(d)$ and $q(d)$ depend only on the length of d. Set $\tilde{q}_s = \tilde{q}(d)$, $q_s = q(d)$ for d short, and $\tilde{q}_l = \tilde{q}(d)$, $q_l = q(d)$ for d long. Let m_s (resp., m_l) be the number of short (resp., long) roots in Σ^+.

Now let Δ be a system of simple roots in Σ and let $w_0 = s_{a_1} \cdots s_{a_m}$, $a_i \in \Delta$, be a reduced expression for w_0. By 3.7 we have $G(k) = G(a_1)(k) \cdots G(a_m)(k)$. Therefore $\alpha(G(k))$ is contained in $G'(a_1) \cdots G'(a_m)$. Since $\alpha(G(k))$ is dense in G' it follows that $G'(a_1) \cdots G'(a_m)$ must contain an open subset of G'. Using 5.2(iii) and the above discussion we see that the sets $V = V(a_1) \cdots V(a_m)$ and $\tilde{V} = V(a_1) \cdots V(a_{m-1})\, \tilde{V}(a_m)$ must contain open subsets of G'. Since $Z_{G'}(T')$ normalizes $U^\pm(d)$ we can write $V = Z_{G'}(T')\, U^+(a_1)\, U^-(a_1) \cdots U^+(a_m)\, U^-(a_m)$ $\tilde{V} = Z_{G'}(T')\, U^+(a_1)\, U^-(a_1) \cdots U^+(a_{m-1})\, U^-(a_{m-1})\, U^-(a_m)\, \tilde{U}^-(a_m)$. We have $\dim V \leqslant q + 2 \bigcup_{i=1}^{m} q(a_i)$, $\dim \tilde{V} \leqslant q + 2 \bigcup_{i=1}^{m-1} q(a_i) + q(a_m) + \tilde{q}(a_m)$.

To continue our argument we need a Lemma.

5.6.1. LEMMA. *Let $w = s_{a_1} \cdots s_{a_p}$ be a reduced expression for $w \in W$. Set $\Sigma_w = \{a \in \Sigma^+ \mid wq \in -\Sigma^+\}$. Let p_s (resp., p_l) be the number of short (resp., long) roots among $a_1, \ldots, a_p$. Then p_s (resp., p_l) is the number of short (resp., long) roots in Σ_w.*

Proof. It follows from [4], Ch. VI, no. 1.6, Proposition 17. Namely, let L be the shortest line joining two points in general position of chambers C and wC. Then it intersects walls corresponding to $b_1, \ldots, b_p$ where $\Sigma_w = \{b_1, \ldots, b_p\}$. Now apply Proposition 17(ii), loc. cit.

5.6.2. *Proof of 5.6 continued.* By 5.6.1 applied to w_0 we have $\dim V \leqslant q + 2m_s q_s + 2m_l q_l$. Since by 5.5 we have $\Sigma'(a) \cap \Sigma'(b) = \varnothing$ unless $a = \pm b$, it follows that $\dim G' \geqslant q + 2 \bigcup_{a\in\Sigma^+} q(a) = q + 2m_s q_s + 2m_l q_l$ and inequality is strict unless $\Sigma' = \Sigma_{a\in\Sigma^+} \Sigma'(a)$. Since $\dim V = \dim G'$ it follows that we have equality and therefore 5.6(i) holds.

Since $\dim \tilde{V} = \dim V$ it follows now that $\tilde{q}(a_m) = q(a_m)$. Since any root $d \in \Delta$ can be chosen as a_m in an appropriate reduced expression for w_0 we have $\tilde{q}(d) = q(d)$ for all $d \in \Sigma$. This means that $p(d)(G'(d))$ contains an open subset of $S(d)$. Since $p(d)(G'(d))$ is evidently closed, we have $p(d)(G'(d)) = S(d)$. Therefore $G'(d)$ acts transitively on the complete variety $S(d)$. Consider now the action of $R_u(G'(d))$ on $S(d)$. Denote by X the set of fixed points of $R_u(G'(d))$. Since $R_u(G'(d))$ is solvable and connected it follows that $X \neq \varnothing$. But clearly, $G'(d)$ maps X into itself. Therefore $X = S(d)$, i.e., $R_u(G'(d))$ acts trivially on $S(d)$. It follows that $R_u(G'(d))$ centralizes $G'(\Sigma'(d))$. Since $G'(d) \subseteq Z_{G'}(T') \cdot G'(\Sigma'(d))$ (by 5.2(iii)) it follows that $R_u(G'(d))$ is contained in semi-simple components of $Z_{G'}(T')$ which commute with $G'(\Sigma'(d))$. But then $R_u(G'(d)) \not\subset [G'(d), G'(d)]$. Therefore 5.2(i) implies that $R_u(G'(d)) = 1$.

Now it follows from 5.3(i) that $T' \cdot U^+(d)\, U^-(d)$ is open in $T' \cdot G'(\Sigma'(d)) = G'(d) \cdot T'$. Therefore $\bar{V} = T'U^-(a_1)\, U^-(a_1) \cdots U^-(a_m)\, U^-(a_m)$ whence $\dim T' = \dim Z_{G'}(T')$, i.e., T' is a maximal torus of G'. This concludes the proof of 5.6.

5.7. Lemma. *Let Σ_1, Σ_2 be two proper root subsystems in Σ.*

(i) *The closure $G'(\Sigma_i)$ of $\alpha(G(\Sigma_i)(k))$ in G' is a proper subgroup of G';*

(ii) *$G'(\Sigma_1) = G'(\Sigma_2)$ if and only if $\Sigma_1 = \Sigma_2$.*

Proof. Let Δ_i be a system of simple roots in Σ_i and let $w_{0,i}$ be the longest element in the Weyl group of Σ_i. Write $w_{0,i} = s_{a_{1,i}} \cdots s_{a_{m(i),i}}$, $a_j \in \Delta_i$. Then $G(\Sigma_i)(k) = G(a_{1,i})(k) \cdots G(a_{m(i),i})(k)$. Therefore $\alpha(G(\Sigma_i)(k))$ is contained and dense in $G'(a_{1,i}) \cdots G'(a_{m(i),i})$. Since the latter product is closed it follows that it is a group, whence our both assertions.

5.8. Lemma. $r = 1$ *(i.e., $|\Sigma'(d)| = 2$ for all $d \in \Sigma$).*

Proof. By 5.6, 5.3(i), 5.4 Σ' is a disjoint union of $|\Sigma^+|$ subsystems of type $A_1 \times \cdots \times A_1$ (r times). Let h and h' be Coxeter numbers of Σ and Σ'. Then $|\Sigma| = mh$, $|\Sigma'| = rmh'$. Since $|\Sigma'| = |\Sigma^+| \cdot |A_1 \times \cdots \times A_1| = mhr$ we have $mrh = rmh'$ whence $h = h'$.

Now for every connected Σ we find all Σ' such that $h = h'$, rank $\Sigma' = r \cdot$ rank Σ. We can assume that $r \geqslant 2$. If Σ is of type A_m then $h = m + 1$ and no system of rank $> m$ has the same h. If Σ is of type B_m, C_m, $m \geqslant 2$, and D_{m-1}, $m \geqslant 3$, then $h = 2m$ and the only root system of greater rank with the same h is A_{2m-1}. But then (rank Σ')/(rank Σ) $\notin \mathbf{Z}$. The same applies to the cases E_6 (resp., E_7); then $h = 12$ (resp., $h = 18$) and Σ' can be of type A_{11}, B_6, C_6, D_7 (resp., A_{17}, B_9, C_9, D_{10}) and (rank Σ)/(rank Σ') $\notin \mathbf{Z}$ in all cases. If Σ is of type E_8 then $h = 30$ and Σ' is of type A_{29}, B_{15}, C_{15}, D_{16}. Here only for D_{16} we have (rank Σ')/(rank Σ) $\in \mathbf{Z}$. In this case take a subsystem of type E_7 in Σ. Then the previous argument together with Lemma 5.7 show that the group $G'(\Sigma_1)$ is an almost direct product of two subgroups of type E_7. But this is impossible.

If Σ is of type F_4 then $h = 12$ and Σ' if of type A_5, B_3, C_3, D_4 and none of them will do. If Σ is of type G_2 then $h = 6$ and Σ' is of type A_5, B_3, C_3, D_4. Of those only D_4 has proper rank. But if we take for Σ_1 the subsystem of type A_2 in Σ then the previous argument together with Lemma 5.7 imply that D_4 contains a subgroup of type $A_2 \times A_2$. But this is false.

5.9. Lemma. *Set $\{\pm d'\} = \Sigma(G'(d), T')$, $d' \in \Sigma'$. Then d is a long root if and only if d' is a long root.*

Proof. It is sufficient to consider rank 2 case, i.e. the case when Σ is of type A_2, B_2, G_2. By Lemma 5.8 we have $|\Sigma| = |\Sigma'|$, rank Σ = rank Σ' whence Σ' is of the same type as Σ. So our assertion is vacuously true if Σ is of type A_2. If Σ is of type B_2, then d is long if and only if $Z_G(G(d))$ is non-commutative (because char $k \neq 2$). It follows that $Z_{G'(k')}(G'(d')(k'))$ is also non-commutative. Since char $k' \neq 2$ this implies that d' is long.

If Σ is of type G_2 then let Σ_1 be the subsystem of long roots. The groups $G(a)$, $a \in \Sigma_1$, generate a subgroup of type A_2. It follows from 5.7 that $G'(\Sigma_1)$ is a proper subgroup of G'. Since char $k' \neq 3$ this subgroup pust be the subgroup of type A_2, corresponding to long roots in Σ', whence our assertion.

5.10. Corollary. *Σ' is of the same type as Σ.*

Proof. We know that $|\Sigma| = |\Sigma'|$, rank Σ = rank Σ', and, finally, Σ and Σ' have the same number of long roots. This implies our claim.

5.11. Let $P\Sigma = \Sigma/\{\pm 1\}$, $P\Sigma = \Sigma'/\{\pm 1\}$ be projectivizations of Σ and Σ'. The map $\{\pm d\} \to \{\pm d'\}$ where $\{\pm d'\} = \Sigma(G'(d), T')$, is a map $\alpha^*: P\Sigma \to P\Sigma'$. Because of 5.7, we know that for any subsystem Σ_1 of rank i in Σ we have $\alpha^*(P\Sigma_1) = P\Sigma_1'$ where Σ_1' a subsystem of the same type as Σ_1 (by 5.10). Moreover, if $\bar{\Sigma}$ is a subsystem of Σ and $\tilde{\Sigma}$ is a subsystem of $\bar{\Sigma}$ then for any $a \in \tilde{\Sigma}$ we have $\alpha^*(P\Pi(Pa, P\bar{\Sigma}, P\tilde{\Sigma})) = P\Pi'(\alpha^*(Pa), \alpha^*(P\bar{\Sigma}), \alpha^*(P\tilde{\Sigma}))$. Here $\Pi(a, \bar{\Sigma}, \tilde{\Sigma})$ is defined as follows. First we construct $H_a = Z_{G(\bar{\Sigma})}(Z_{G(\tilde{\Sigma})}(G(a)))$ then we take for $\Pi(Pa, P\bar{\Sigma}, P\tilde{\Sigma})$ the set of $d \in \Sigma(H_a, T)$ such that d has the same length as a itself. The set $\Pi(Pa, P\bar{\Sigma}, P\tilde{\Sigma})$ can be defined in abstract group terms because $H_a(k) = Z_{G(\bar{\Sigma})(k)}(Z_{G(\tilde{\Sigma})(k)}(G(a)(k)))$.

Let us recall that a subsystem $\Sigma_1 \subset \Sigma$ is called saturated if $\mathbf{Q}\Sigma_1 \cap \Sigma = \Sigma_1$. We are going to prove the following

5.11.1. Proposition. *Let $\alpha^*: P\Sigma \to P\Sigma'$ be a map which satisfies the following properties*

(i) *If Σ_1 is a saturated subsystem of Σ then $\alpha^*(P\Sigma_1) = P\Sigma_1'$ where Σ_1' is an isomorphic saturated subsystem of Σ';*

(ii) *If $d \in \Sigma$ and $\alpha^*(\{\pm d\}) = \{\pm d'\}$ then d and d' are simultaneously long or short;*

(iii) *If $\bar{\Sigma}$ is a saturated subsystem of Σ and $\tilde{\Sigma}$ is a proper saturated subsystem of $\bar{\Sigma}$ of maximal rank then for any $a \in \bar{\Sigma} - \tilde{\Sigma}$ we have $\alpha^*(P\Pi(Pa, P\bar{\Sigma}, P\tilde{\Sigma})) = P\Pi'(\alpha^*(Pa), \alpha^*(P\bar{\Sigma}), \alpha^*(P\tilde{\Sigma}))$.*

Then there exists an isomorphism $\tilde{\alpha}: \Sigma \to \Sigma'$ such that $\alpha^*(\{\pm a\}) = \pm\tilde{\alpha}(a)$ for any $a \in \Sigma$. Moreover, $\tilde{\alpha}$ is determined uniquely up to sign.

The proof uses case considerations and will be given in a number of steps. We use numeration of roots given in Tables of [4]. For $d \in \Sigma$ and $d' \in \Sigma'$ we set $\bar{d} = \{\pm d\}$, $\bar{d}' = \{\pm d'\}$.

5.11.2. Lemma. *Proposition 5.11.1 holds for root systems of rank 2.*

Proof. If Σ is of type A_2 then any bijection $P\Sigma \to P\Sigma'$ is induced by an isomorphism $\Sigma \to \Sigma'$ which is unique up to sign (because the Weyl group is the symmetric group on 3 letters and $|P\Sigma| = 3$). If Σ is of type B_2 then

$$\alpha^*(\bar{a}_1 \cup \overline{a_1 + 2a_2}) = \bar{a}_1' \cup \overline{a_1' + 2a_2'}\,.$$

By changing a choice of simple root system we can assume that $\alpha^*(\bar{a}_1) = \bar{a}_1'$. $\alpha^*(\overline{a_1 + 2a_2}) = \overline{a_1' + 2a_2'}$. Now we have $\alpha^*(\bar{a}_2) = \bar{a}_2'$ or $\alpha^*(\bar{a}_2) = \overline{a_1' + a_2'}$. These two cases are permuted by the reflection in a_1 (which does not affect our previous choices. So we can assume that $\alpha^*(\bar{a}_2) = \bar{a}_2'$, i.e., α^* is induced by $\tilde{\alpha}$ given by $\tilde{\alpha}(a_i) = a_i'$.

If Σ is of type G_2 then the sybsystem Σ_1 of long roots is of type A_2 and therefore the restriction of α^* to $P\Sigma_1$ is induced by an isomorphism $\tilde{\alpha}: \Sigma_1 \to \Sigma_1'$. For $d \in \Sigma_1$ we have (by condition 5.11.1 (iii)) $\alpha^*(\bar{d}^{\perp}) = \overline{\tilde{\alpha}(d)}^{\perp}$ where $d^{\perp}$ is the positive root orthogonal to d (orders on Σ and Σ' are assumed to be respected by $\tilde{\alpha}$). So setting $\tilde{\alpha}(d^{\perp}) = \tilde{\alpha}(d)^{\perp}$ we obtain a desired isomorphism.

5.11.3. Let us proceed by induction. First we identify Σ and Σ' so that $\alpha^*: P\Sigma \to P\Sigma$. Next we assume that Proposition 5.11.1 holds for any proper saturated subsystem and apply this assumption to a specially chosen connected saturated sybsystem $\tilde{\Sigma}$ of maximal rank (the description of $\tilde{\Sigma}$ is explicitly given below). Then we can assume that $\alpha^*: P\tilde{\Sigma} \to P\tilde{\Sigma}$ (applying if necessary an automorphism of Σ) and because of our induction assumptions we can assume that there exists $\tilde{\alpha}: \tilde{\Sigma} \to \tilde{\Sigma}$ such that $\alpha^*(\bar{a}) = \overline{\tilde{\alpha}(a)}$ for $a \in \tilde{\Sigma}$. Then we can assume that $\tilde{\alpha}: \tilde{\Sigma} \to \tilde{\Sigma}$ is the identity map.

Let Δ be a subsystem of simple roots in Σ such that $\tilde{\Delta} = \tilde{\Sigma} \cap \Delta$ is a subsystem of simple roots in $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is of maximal rank we have $|\Delta - \tilde{\Delta}| = 1$. Let $a \in \Delta - \tilde{\Delta}$. Since $\tilde{\alpha} = id$ on $\tilde{\Sigma}$ we see that condition 5.11(iii) becomes $\alpha^*(a) \in P\Pi(Pa, P\Sigma, P\tilde{\Sigma})$. Then (with our choice of $\tilde{\Sigma}$) it turns out that in all cases but F_4 we have $|P\Pi(Pa, P\Sigma, P\tilde{\Sigma})| = 1$ or 2. If $|P\Pi(Pa, P\Sigma, P\tilde{\Sigma})| = 1$ then it means that $\alpha^*(\bar{a}) = \bar{a}$ and setting $\tilde{\alpha}(a) = a$ we get the required map. If $|P\Pi(Pa, P\Sigma, P\tilde{\Sigma})| = 2$ then $\alpha^*(\bar{a}) = \bar{a}$ or $\bar{b}$, where $\bar{b}$ is another element of

$P\Pi(Pa, P\Sigma, P\tilde{\Sigma})$. If $\alpha^*(\bar{a}) = \bar{a}$ we set $\tilde{\alpha}(a) = a$ and we are done. If $\alpha^*(\bar{a}) = \bar{b}$ then it turns out (it is checked explicitly) that there exists an element w of the Weyl group of Σ such that $w\tilde{\Sigma} = \tilde{\Sigma}$, $w = -id$ on $\tilde{\Sigma}$, $w\bar{a} = \bar{b}$. So we choose $\tilde{\alpha}$ by setting it to be -1 on $\tilde{\Sigma}$ and setting $\tilde{\alpha}(a) = wa$. Then we show that such definitions do in fact satisfy $\alpha^*(\bar{d}) = \overline{\tilde{\alpha}(d)}$ for all $d \in \Sigma$.

Let us now make the necessary verifications. We take a system of simple roots Δ in Σ and take for a an extremal weight a_i of the Dynkin diagram (the number in the numeration of Tables of [4] will be specified below). Then we set $\tilde{\Delta} = \Delta - a$, $\tilde{\Sigma} = \mathbf{Q}\tilde{\Delta} \cap \Sigma$. We write $\Pi(a)$ for $P\Pi(Pa, P\Sigma, P\tilde{\Sigma})$ and describe $\Pi(a)$ by writing down positive roots b such that $\bar{b} \in \Pi(a)$.

If Σ is of type A_{m-1}, take $a = a_1$, then $\Pi(a) = \{\bar{a}\}$. If Σ is of type B_{m+1}, take $a = a_1$, then $\Pi(a) = \{\bar{a}, \bar{b}\}$, $b = a_1 + 2\sum_{i>1} a_i$. If Σ is of type C_{m+1}, take $a = a_1$, then $\Pi(a) = \{\bar{a}\}$. If Σ is of type D_{m+1}, take $a = a_1$ then $\Pi(a) = \{\bar{a}, \bar{b}\}$, $b = a_1 + 2\sum_{i=2}^{m-1} a_i + a_m - a_{m+1}$. If Σ is of type E_6, E_7, E_8, take $a = a_1, a_7, a_1$ respectively, then $\Pi(a) = \{\bar{a}\}$.

We recall that the case F_4 will be dealt with later. Let us now show that the above constructions do indeed give us the required $\tilde{\alpha}$ (i.e., that $\alpha^*(\bar{d}) = \overline{\tilde{\alpha}(d)}$ for all $d \in \Sigma$). This follows from our result for rank 2 case (cf. 5.11.2) and the following Lemma (which we apply with $\Sigma_1 = (\mathbf{Q}a + \mathbf{Q}c) \cap \Sigma$ where c is the unique neighbour of a in the Dynkin diagram of Σ).

5.11.4. Let Σ be a root system, let Δ be its subsystem of simple roots. Let Σ^+ be the corresponding system of positive roots. For any subsystem $\bar{\Sigma}$ such that $\bar{\Sigma} = -\bar{\Sigma}$ we put $\bar{\Sigma}^- = \bar{\Sigma} \cap \Sigma^+$ and we denote by $\Delta(\bar{\Sigma})$ the system of simple roots in $\bar{\Sigma}$ which corresponds to $\bar{\Sigma}^+$. We take $\tilde{\Delta} \subset \Delta$ with $|\Delta - \tilde{\Delta}| = 1$ and such that $\tilde{\Delta}$ is connected and we set $\tilde{\Sigma} = \mathbf{Q}\tilde{\Delta} \cap \Sigma$. Let $\Delta_1 \subset \Delta$, $|\Delta_1| = 2$, $|\Delta_1 \cap \tilde{\Delta}| = 1$, $\Sigma_1 = \mathbf{Q}\Delta_1 \cap \Sigma$. Assume that Δ_1 is connected.

LEMMA. *There exist connected saturated subsystems $\Sigma_2, \dots, \Sigma_m$ of rank 2 in Σ such that*

$$\text{(a)} \qquad \Delta(\Sigma_i) \subset \Big(\bigcup_{j<i} \Sigma_j\Big) \cup \tilde{\Sigma}, \qquad \textit{for} \quad i > 1,$$

$$\text{(b)} \qquad \Sigma_i \not\subset \Big(\bigcup_{j<i} \Sigma_j\Big) \cup \tilde{\Sigma}, \qquad \textit{for} \quad i > 1,$$

$$\text{(c)} \qquad \Sigma = \Big(\bigcup_{j\leqslant m} \Sigma_j\Big) \cup \tilde{\Sigma}.$$

Proof. Suppose that we have already constructed $\Sigma_2, \dots, \Sigma_q$ such that $(\bigcup_{j\leqslant q} \Sigma_j) \cup \tilde{\Sigma} \neq \Sigma$. Let a be the smallest root in $\Sigma^+ - [(\bigcup_{j\leqslant q} \Sigma_j^-) \cup \tilde{\Sigma}^+]$ then there exist $b_1 \in \Delta$ and $b_2 \in \Sigma^-$ such that $b_1 + b_2 = a$. Therefore $b_2 < a$, whence $b_2 \in (\bigcup_{j\leqslant q} \Sigma_j^+) \cup \tilde{\Sigma}^-$. The same inclusion holds for b_1 since $b_1 \in \Delta \subset \Sigma_1^+ \cup \tilde{\Delta}$.

Set $\Sigma_{q+1} = (\mathbf{Q}b_1 - \mathbf{Q}b_2) \cap \Sigma$. Then, by construction, Σ_{q+1} satisfies (a) and (b), as required.

5.11.5. Let us conclude the proof of 5.11.1 by

LEMMA. *Proposition* 5.11.1 *holds for type* F_4.

Proof. Let Σ be a system of type F_4, $\alpha^*: P\Sigma \to P\Sigma$ a map of 5.11.1. Let Σ_1 be the subsystem of long roots. We have $\alpha^*(P\Sigma_1) = P\Sigma_1$ and since Σ_1 is of type D_4 there exists $\tilde{\alpha}_1: \Sigma_1 \to \Sigma_1$ such that $\alpha^\times(\bar{a}) = \overline{\tilde{\alpha}_1(a)}$ for $a \in \Sigma_1$. We can assume that $\tilde{\alpha}_1 = id$. Then for $d \in \Sigma - \Sigma_1$ we have $P\Pi(Pd, \Sigma, \Sigma_1) = \{\bar{d}\}$ whence it follows that setting $\tilde{\alpha}(d) = d$ we get a desired map. Its unicity follows from unicity of $\tilde{\alpha}_1$.

6. Homomorphisms of Admissible Groups: Proof of the Main Theorem

In this section we conclude the proof of the Main Theorem, stated in the Introduction. Actually, what is left of the proof is completely formal and straightforward. We continue to use notation of the preceding section.

6.1. For every $d \in \Sigma$ let $\varphi(d)$, $A(d)$, $I(d)$, $\beta(d)$ be the data obtained from the homomorphism $\alpha: G(d)(k) \to G'(\tilde{\alpha}(d))(k') = G'(d)(k')$ as described in Theorem 4.1.

LEMMA. $\varphi(d)$ *does not depend on* $d \in \Sigma$.

Proof. Clearly, d depends only on length of d (because roots of the same length are permuted by the Weyl group of Σ and because of 3.2). Let $\tilde{\Sigma}$ be the subsystem of long roots of Σ. Since the rank of $\tilde{\Sigma}$ is the same as the rank of Σ, the above together with 3.7.1 shows that we have a map $\varphi^0: T(k) \to {}^{\varphi}T(k')$ where $\varphi = \varphi(d)$ for $d \in \tilde{\Sigma}$.

In the basis e_1, e_2, e_3 constructed in the proof of 4.6 (the underlying quadratic form is $x_1^2 + x_2^2 + x_3^2$ in this basis) the elements from $T(d)(k)$, $d \in \tilde{\Sigma}$, have the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $a^2 + b^2 = 1$, $a, b \in k$. And the action of $\varphi(d)$ is given in this basis by $\varphi\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varphi(a) & \varphi(b) \\ -\varphi(b) & \varphi(a) \end{smallmatrix}\right)$. Let Φ be an extension of φ to the quadratic extension K of k; Φ is unique since the integral closure $A(-1^{1/2})$ of A in K is unramified over A. We can write $\Phi(a + (-1)^{1/2}b) = \varphi(a) + (-1)^{1/2}\varphi(b)$ whence we see that $\varphi(d)$ acts on $T(d)(k)$ as Φ on $K^1 = \mathrm{Ker}(N_{K/k}: K^\times \to k^\times)$. Since $T(k)$ is the direct product of $T(d)$, where d runs over a subsystem of simple roots in $\tilde{\Sigma}$, the same is true for $T(k)$. That is, Φ acts on $T(k)$ as the restriction of Φ from $T(K)$.

Let now d be a short root. Then in the same way as above we find that $\varphi(d)$ acts on $T(d)(k)$ as the restriction of its extension $\Phi(d)$ to K. To continue our argument we need the following

6.1.1. Lemma. *Let Φ_1, Φ_2 be two places of K which coincide on $K^1 = \mathrm{Ker}(N_{K/k}: K^* \to k^*)$. Then they coincide.*

Proof. The valuation rings of Φ_1 and Φ_2 both contain $\mathbf{Q}$. Therefore Φ_1 and Φ_2 are trivial on $\mathbf{Q}$ and, in particular, they coincide on $\mathbf{Q}$. Therefore they coincide on $\mathbf{Q}((-1)^{1/2})$ too. Next we have $\Phi_i(a-(-1)^{1/2}b) = \varphi_i(a) + (-1)^{1/2}\varphi_i(b)$ where φ_i is the restriction of Φ_i to k. Now $\Phi_1((a+(-1)^{1/2}b) + (a-(-1)^{1/2}b)) = \Phi_2((a+(-1)^{1/2}b) + (a-(-1)^{1/2}b)) = \varphi_1(2a) = \varphi_2(2a)$. Since Φ_i coincide on $\mathbf{Q}((-1)^{1/2})$ it follows that $\varphi_1(a) = \varphi_2(a)$, $\varphi_1(b) = \varphi_2(b)$ whenever $a^2 + b^2 = 1$. Now we look at our construction of φ from a projective plane. We recall that because of 1.2(iii) $T(d)(k)$ acts transitively on the set of lines through $T(d)(k)$, which means that K^1 acts transitively on $\mathbf{P}^1(k)$. This action is given by $x \to (ax - b)/(-bx + a)$ where $x \in k \subset \mathbf{P}^1(k) = k \cup \infty$. Because of transitivity we have: $k \cup \infty = \{(a-b)/(-b+a) \mid a, b \in k, a^2 + b^2 = 1\}$ (we take $x = 1$). Because of the above remarks it follows that Φ_i coincide on k. Since $A((-1)^{1/2})$ is unramified over A we have $\Phi_1 = \Phi_2$ as claimed.

6.1.2. Proof of 6.1 continued. Now we again consider our short root d. We can assume that there exist two long roots d_1, d_2 such that $\Sigma_1 = (\mathbf{Q}d_1 + \mathbf{Q}d_2) \cap \Sigma$ is a connected rank 2 system and $d \in \Sigma_1$. Expressing d through d_1, d_2 and taking 6.1.1 into account we obtain a relation $\Phi(d)^n = \Phi^m$ which implies (because char $k' = 0$) that $\Phi(d) = \Phi$ as asserted.

6.2. Now we construct on G a structure of a group scheme over A.

Lemma. *There exists on G a unique structure of a semi-simple group scheme G_A over A such that for every admissible torus T and for every $d \in \Sigma(G, T)$ the A-structure on $G(d)$ determined according to* 4.1 *from the restriction of α to $G(d)$ agrees with the structure induced by G_A.*

Proof. Let us first prove existence. Unicity will be evident. Fix an admissible torus T and set $\Sigma = \Sigma(G, T)$. Let Δ be a system of simple roots in Σ. For every $d \in \Delta$ we have by 4.1 a structure $G(d)_A$ of a semi-simple group scheme over A on $G(d)$. This gives us an A-scheme structure on $T(d) = T \cap G(d)$, $d \in \Delta$. Since T is the direct product of $T(d)$, $d \in \Delta$, we get an A-structure on T. Let $x_d: \mathbf{G}_{a,K} \to G$, $d \in \Sigma$, be K-homomorphisms which give a Chevalley parametrization of root subgroups. We can assume that for $d \in \pm\Delta$ the maps $x_d: \mathbf{G}_{a,A((-1)^{1/2})} \to G(d)$ give a Chevalley parametrization for $G(d)_{A((-1)^{1/2})}$. This defines a Chevalley group scheme $G_{A(-1^{1/2})}$ over $A((-1)^{1/2})$ such that $G_{A(-1^{1/2})} \otimes K = G_-$. Now use Appendix 2, Theorem 8.8 to define a group scheme G_A over A by setting $d_{a,\sigma} = -1$ for all $a \in \Sigma$ where $\sigma x = \bar{x}$ for $x \in K$. This group scheme is semi-simple (since it is a form of a Chevalley group scheme). Clearly we have $G_k = G_A \otimes k$. For $d \in \Delta$ the restriction of our structure to $G(d)$ gives us $G(d)_A$ (by 4.6) and we have $G(d)_A(A) = G(d)(k)$ (by 4.1(ii)). Since any system of simple roots is conjugate to Δ by an element of the

Weyl group and the latter is generated by its intersection with the $G(d)_A(A)$, $d \in \Delta$, it follows that our construction does not depend on the choice of Δ. By 3.7 and 3.5 it does not depend on the choice of an admissible torus T. This proves the existence. The proof also shows that our A-structure is uniquelly determined by the A-structures on $G(d)$, $d \in \Delta$, whence the unicity.

6.3. Now we show that G' is actually defined over $\varphi(A)$. For convenience we set $k'' = \varphi(A)$.

LEMMA. *There exist on G' a unique structure of a group defined over k'' such that $\alpha(G(k)) \subseteq G'(k'')$.*

Proof. The proof of the above Lemma follows the same lines as the proof of 6.2. Let T be an admissible torus of G and let T' be the Zariski-closure of $\alpha(T(k))$. For every $d \in \Sigma(G, T)$ the closure $G'(d)$ of $\alpha(G(d)(k))$ is defined over k'' (by 4.1(iii)). In particular, $T'(d) = T' \cap G'(d)$ is defined over k''. This makes T' into a k''-torus since T' is a product of $T'(d)$, $d \in \Delta$, Δ a simple root system of $\Sigma(G, T)$. Now we choose $x'_{\tilde{\alpha}(d)}: \mathbf{G}_{a,k''((-1)^{1/2})} \to G'(d)$, $\pm d \in \Delta$, to be a Chevalley parametrization in $G'(d)$, $d \in \Delta$. This choice makes $G'_{k''((-1)^{1/2})}$ into a Chevalley group over $k''((-1)^{1/2})$. Now define the action of $\mathrm{Gal}(k''((-1)^{1/2})/k'')$ by $d_{a,\sigma} = -1$ (cf. Appendix 2, 8.8, 8.9). This makes G' into a k''-group and agrees with the structures on $G'(d)$ given by 4.1(iii), 4.5.2. Since the Weyl group acts transitively on the set of simple root systems and is generated by its intersection with the $G'(d)$, $d \in \Delta$, it follows that the result of our construction does not depend on the choice of Δ in $\Sigma(G, T)$. By 3.7, 3.5 it also does not depend on the choice of an admissible T. This proves the existence. The proof also shows that our k''-structure on G' is uniquelly determined by ones on $G(d)$, $d \in \Delta$, whence the unicity.

6.4. Finally we must construct $\beta: {}^{\varphi}G_A \to G'_{\varphi(A)}$.

LEMMA. *There exists a unique central $\varphi(A)$-isogeny $\beta: {}^{\varphi}G_A \to G'_{\varphi(A)}$ such that for any admissible torus T and every $d \in \Sigma(G, T)$, the restriction $\beta(d)$ of β to $G(d)$ is the map β of* 4.1(iv) *constructed from the restriction of α to $G(d)(k)$.*

Proof. We can assume without loss of generality that $k'' = \varphi(A)$ (otherwise replace k' by $\varphi(A)$). Let T be an admissible torus of G, let T' be the Zariski closure of $\alpha(T(k))$ and let $\tilde{\alpha}: \Sigma(G, T) \to \Sigma(G, T)$ be an isomorphism of 5.1(iii). Let Δ be a simple root system in $\Sigma(G, T)$. Then $\Delta' = \tilde{\alpha}(\Delta)$ is a simple root system in $\Sigma(G', T')$. We denote $\beta(d)$ the β given by 4.1(iv) from the restriction of α to $G(d)(k)$, $d \in \Sigma(G, T)$. So $\beta(d): {}^{\varphi}G(d)_A \to G'_{k'}$. We choose a Chevalley parametrizations $x_d: \mathbf{G}_{a,A((-1)^{1/2})} \to G_{A((-1)^{1/2})}$ of G_A over $A((-1)^{1/2})$. Next we choose a Chevalley parametrization $x_d: \mathbf{G}_{a,k'((-1)^{1/2})} \to G'$ of $G'_{k'}$ over $k'((-1)^{1/2})$. We can assume without any loss of generality that $\beta(d) \circ \varphi^0 \circ x_d = x'_{\epsilon(d)\tilde{\alpha}(d)}$ where $\epsilon(d): \Delta \to \{\pm 1\}$. We can assume (replacing if necessary $\tilde{\alpha}$ by $-\tilde{\alpha}$) that $\epsilon(d_0) = 1$ for at least one $d_0 \in \Delta$. Now we define β by $\beta \circ \varphi^0 \circ x_d = x'_{\tilde{\alpha}(d)}$, $d \in \Delta$. This defines a central k'-isogeny ${}^{\varphi}G_A \to G_k$ because $d_{a,\sigma} = d_{\bar{\alpha}(a),\sigma} = -1$ for all $a \in \Sigma$ (cf. Appendix 2, Lemma 8.7). We want to prove that β coincides with $\beta(d)$ for $d \in \Sigma$. This is true by our choice of d_0 for $\beta(d_0)$. If $\beta(d) \neq \beta \mid {}^{\varphi}G(d)$, then $\beta(d)(\varphi^0(t)) = \beta(\varphi^0(t^{-1}))$ for $t \in T(d)(k)$ (since in this case the effect of $\beta(d)$ and β differ by a conjugation by $n \in N_{G'(d)}(T'(d))$.

Now if $n \in N_{G(d)}(T(d))$, $d \in \Delta$, then $\beta(d)(n) \notin T'$ and $\beta(n) \notin T'$ and since $N_{G(d)}(T(d))/T(d) = \mathbf{Z}/2$, we have that $\beta(d)(n) \in \beta(n) \cdot T'$. If $n \in N_{G(d)(k)}(T(d)(k)$ then it means that $\alpha(n) \in \beta(n) \cdot T'$. Since such the n generate the Weyl group of G (when d varies over Δ) we see that $\alpha(n) \in \beta(n) \cdot T'$ for any $n \in N_{G(k)}(T(k))$. Now $\alpha(ntn^{-1}) = \beta(ntn^{-1})$ if $t \in T(d_0)(k)$, $n \in N_{G(k)}(T(k))$, by our choice of d_0. This means that $\alpha(t) = \beta(t)$ for all $t \in T(k)$. It follows that $\beta(d)(\varphi^0(t)) = \beta(\varphi^0(t))$ for all $t \in T(d)(k)$ (by the construction of $\beta(d)$, cf. 4.1(iv)). This means that $\beta(d) = \beta$ for all $d \in \Delta$. Now it follows from conjugacy of tori and from transitivity of the Weyl group on the set of systems of simple roots that our Lemma holds.

6.5. We can conclude our proof with the

LEMMA. *For β from Lemma 6.4 we have $\alpha(g) = \beta(\varphi^0(g))$ for all $g \in G(k)$.*

Proof. We have $\alpha(g) = \beta(d)(\varphi^0(g)$ for $g \in G(d)(k)$, $d \in \Delta$. By 6.4 it implies that $\alpha(g) = \beta(\varphi^0(g))$ for $g \in G(d)(k)$, $d \in \Delta$. A reference to 3.7 concludes the proof.

7. APPENDIX 1. A THEOREM ABOUT HOMOMORPHISMS OF PROJECTIVE SPACES

For convenience of reference we reprove below a theorem of W. Klingenberg [8] and then restate it in the form we are going to use.

7.1. THEOREM. *Let k and k' be fields and $\bar{\alpha}$: $\mathbf{P}^2(k) \to \mathbf{P}^2(k')$ be a homomorphism of projective planes (i.e., a mapping which maps lines into lines and such that $\bar{\alpha}(\mathbf{P}^2(k))$ is not contained in a line). Let V and V' be the underlying vector spaces of $\mathbf{P}^2(k)$ and $\mathbf{P}^2(k')$. Then*

(i) *there exists a unique place φ: $k \to (k' \cup \infty)$; let $A = \{a \in k \mid \varphi(a) \neq \infty\}$ be the valuation ring of φ,*

(ii) *there exists a unique (up to multiplication by a scalar from k) free A-submodule M of V,*

(iii) *there exists a unique (up to multiplication by a scalar from k'^*) additive map $\bar{\beta}$: $M \to V'$ satisfying $\bar{\beta}(am) = \varphi(a)\,\bar{\beta}(m)$ for $m \in M$, $a \in A$, such that for any line $L \subset V$ one has $\bar{\alpha}(L) = \bar{\beta}(M \cap L)$.*

7.1.1. *Remark.* The proof is the same as the standard proof of the Fundamental Theorem of Projective Geometry.

Notation. L stands for a line, $[a, b]$ stands for a line passing through points a, b.

7.2. But before giving a proof of the Theorem let us show that its converse is also true.

PROPOSITION. *Let k and k' be two fields and let $\varphi: k \to k'$ be a place with valuation ring A. Let V and V' be vector spaces over k and k' respectively and let M be a free A-submodule of V. Let further $\bar\beta: M \to V'$ be an additive map such that $\bar\beta(am) = \varphi(a)\,\bar\beta(m)$. Set $\bar\alpha(L) = \bar\beta(L \cap M)$ for a line $L \subset V$. Then $\bar\alpha$ is a homomorphism of $\mathbf{P}^2(k)$ to $\mathbf{P}^2(k')$.*

Proof. Let I be the maximal ideal of A. First we have to show that for any line $L = kx$, $x \in V$, $x \neq 0$, the A/I-module $L \cap M/L \cap IM$ is free on one generator (that will mean that $\bar\alpha$ is a map $\mathbf{P}^2(k) \to \mathbf{P}^2(k')$). Let e_1, e_2, e_3 be a basis of M. Write $x = ae_1 + be_2 + ce_3$ with $a, b, c \in k$. We can assume that $a, b, c \in A$. Then one of a, b, c divides the other two (cf. [3, Chap. VI, Sect. 1, Theorem 1(d)]. Say it is a. Then replacing x by $a^{-1}x$ we can assume that $x = e_1 + ae_2 + be_3$, $a, b \in A$. Therefore $L \cap M/L \cap IM \neq 0$. Let $y \in L \cap M$. Then $y = de_1 + dae_2 + dbe_3$, $d \in k$. Since $y \in M$ we have $d \in A$ and therefore $y + IM \subset Ax + IM$, i.e., $L \cap M/L \cap IM$ is one-dimensional over A/I.

Now let $N = kx + ky$ be a two-dimensional subspace of V (i.e., a line of $\mathbf{P}^2(k)$). Let us show that $N \cap M/N \cap IM$ is two-dimensional over A/I. (This will mean that lines of $\mathbf{P}^2(k)$ are mapped into lines of $\mathbf{P}^2(k')$.) As above we write (without loss of generality) $x = e_1 + ae_2 + be_3$ and we can assume now that $y = ce_2 + de_3$ with $c, d \in A$. Then either c divides d or d divides c [3, Chap. VI, Sect. 1, Theorem 1(d)]. Suppose c divides d. Then we can assume $y = e_2 + de_3$. This shows that $\dim_{A/I}(N \cap M/N \cap IM) \geqslant 2$. Let $z \in N \cap M$. Then $z = sx + ty$, $s, t \in k$. Since M is free we have $s, t \in A$, whence the dimension of our quotient is exactly 2. This concludes the proof.

7.2.1. *Remark.* It should be noted that we have actually shown (without assuming it) that the image of a line of $\mathbf{P}^2(k)$ is never a point of $\mathbf{P}^2(k')$.

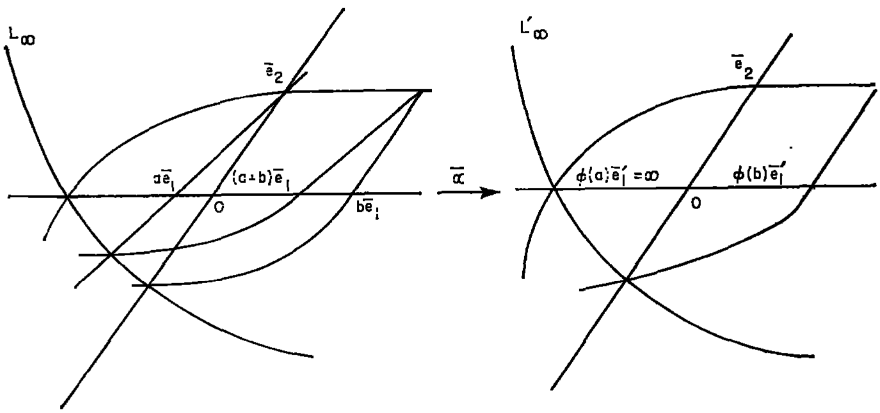
7.3. Now let us pass to the proof of the Theorem. But first let us remark that the only point which is not contained in the standard proofs is the unicity of lattice M.

7.3.1. We follow [6, Chap. III, Sect. 1] and refer to drawings there. First we note that $\bar\alpha(\mathbf{P}^2(k))$ is a subplane of $\mathbf{P}^2(k')$ which is automatically isomorphic (cf. for example [18],) to $\mathbf{P}^2(k'')$ for some subfield $k'' \subseteq k''$. We may (and shall) assume that $k' = k''$. Let e_1', e_2', e_3' be a basis of V'. Choose a basis e_1, e_2, e_3 of V such that $\bar\alpha(ke_i) = k'e_i'$ and $\bar\alpha(k(e_1 + e_3)) = k'(e_1' + e_2')$, $\bar\alpha(k(e_2 + e_3)) = k'(e_2' + e_3')$. Introduce $\varphi: k \to k' \cup \infty$ by $\bar\alpha(k(ae_1 + e_3)) = k'(\varphi(a)e_1' + e_3')$. We have $\varphi(0) = 0'$, $\varphi(1) = 1'$. Let $L_\infty' = ke_1' + ke_2'$ be the infinite line of $\mathbf{P}^2(k')$. We identify subset $\tilde V = k(ke_1 + ke_2 + e_3)$ of $\mathbf{P}^2(k)$ with a two-dimensional vector space over k considering $k(ae_1 + be_2 + e_3)$ as a vector $a\bar e_1 + b\bar e_2$.

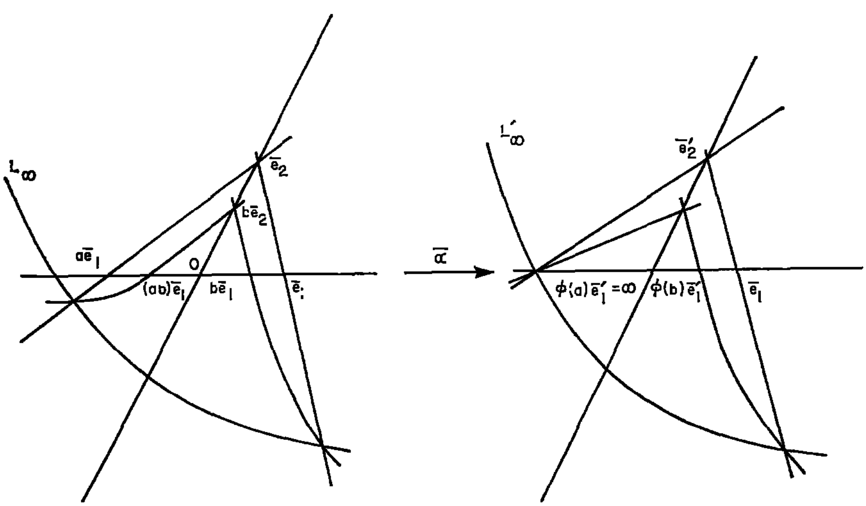
Similarly, we identify $\tilde{V}' = k'(k'e_1 + k'e_2 + e_3)$ with a two-dimensional vector space over k'.

7.3.2. LEMMA. *$\varphi: k \to k' \cup \infty$ is a place of k.*

Proof. Let $a, b \in k$ and suppose that $\varphi(a) \neq \infty$, $\varphi(b) \neq \infty$. Then the reasoning and pictures in [6, Chap. III, Sect. 1] show that $\varphi(a + b) = \varphi(a) + \varphi(b)$, $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ (since constructions describing $a + b$ and ab in $\mathbf{P}^2(k)$ are mapped by $\bar{\alpha}$ on the corresponding constructions in $\mathbf{P}^2(k')$). The same argument shows that if $\varphi(a) = \infty$, $\varphi(b) \neq \infty$ then $\varphi(a + b) = \infty$



as can be seen from the above picture. It also shows that $\varphi(-a) = \infty$ (if $\varphi(a) = \infty$). If $\varphi(a) = \infty$ and $\varphi(b) \neq 0$ then we have the following picture



whence $\varphi(ab) = \infty$. The similar picture shows also that $\varphi(a^{-1}) = 0$ if $\varphi(a) = \infty$ and $\varphi(a^{-1}) = \infty$ if $\varphi(a) = 0$. This concludes the proof of the Lemma.

7.3.3. LEMMA. *We have* $\bar{\alpha}(b\bar{e}_2) = \varphi(b)\bar{e}_2'$.

Proof. The assertion is already proved if $b = 0$. Since the lines parallel to $[\bar{e}_1, \bar{e}_2]$ go under $\bar{\alpha}$ into lines parallel to $[\bar{e}_1', \bar{e}_2']$ we have $\bar{\alpha}(b\bar{e}_2) = \varphi(b)\bar{e}_2'$ as asserted.

7.3.4. LEMMA. (i) *If* $\bar{\alpha}(a\bar{e}_1 + b\bar{e}_2) \notin L_\infty'$ *then* $\bar{\alpha}(a\bar{e}_1 + b\bar{e}_2) = \varphi(a)\,\bar{e}_1 + \varphi(b)\,\bar{e}_2$.

(ii) $\tilde{M} = \{a\bar{e}_1 + b\bar{e}_2 \mid \bar{\alpha}(a\bar{e}_1 + b\bar{e}_2) \notin L_\infty'\}$ *is a free* A*-module (where* $A = \{a \in k \mid \varphi(a) \neq \infty\}$) *with basis* $\bar{e}_1, \bar{e}_2$.

(iii) $\bar{\alpha}\colon \tilde{M} \to \tilde{V}'$ *is an additive map satisfying* $\bar{\alpha}(bm) = \varphi(b)\,\bar{\alpha}(m)$ *for* $m \in \tilde{M}$, $b \in A$.

Proof. Since the point $a\bar{e}_1 + b\bar{e}_2$ can be obtained as an intersection of lines parallel to $[0, e_2]$ and $[0, e_1]$ through $a\bar{e}_1$ and $b\bar{e}_2$ respectively, and since this construction is mapped by $\bar{\alpha}$ onto a similar construction in $\tilde{V}'$, it follows that all our operations may be considered in coordinates and then our lemma is obvious.

Now set $R = \{ae_1 + be_2 + e_3 \mid a, b \in A\}$ and let M be the A-submodule of V generated by R. Define $\bar{\beta}\colon M \to V'$ by $\bar{\beta}(\Sigma a_i e_i) = \Sigma\varphi(a_i)\, e_i'$.

7.3.5. LEMMA. *M is a free A-module with basis* e_1, e_2, e_3.

Proof. M contains $e_1 + e_3, e_2 + e_3, e_3$. Therefore $M \supset Ae_1 + Ae_2 + Ae_3$. But by construction $M \subset Ae_1 + Ar_2 + Ae_3$, whence our assertion.

7.3.6. LEMMA. *For any line* $L = kx \subset V$, $x \neq 0$, *we have* $\bar{\alpha}(L) = \bar{\beta}(L \cap M)$.

Proof. We already proved (cf. 3.4(i)) the above assertion for lines L such that $L \cap M \subset A^*(A_1 + Ae_2 + e_3)$. To show that the same holds for all lines we cover, as usual, $\mathbf{P}^2(k)$ by three affine planes: $\tilde{V} = k(ke_1 + ke_2 + e_3)$, $\tilde{V}_1 = k(e_1 + ke_2 + ke_3)$, $\tilde{V}_2 = k(ke_1 + e_2 + ke_3)$, then we proceed with construction of $\varphi, \beta, R, \tilde{M}$ for each of those and then check that the results coincide.

Let us take $\tilde{V}_1$. We know that $\bar{\alpha}(k(e_1 + e_3)) = k'(e_1' + e_3')$. Let us show that $\bar{\alpha}(k(e_1 + e_2)) = k'(e_1' + e_2')$. We consider $k(e_1 + e_2)$ as the intersection of planes $ke_1 + ke_2$ and $k(e_1 + e_3) + k(e_2 - e_3)$. The image of these planes under $\bar{\alpha}$ is $k'e_1' + k'e_2'$ and $k'(e_1' + e_3') + k'(e_2' - e_3')$ (because of choice of the e_i in 7.3.1 and because $\bar{\alpha}(-\bar{e}_2) = \bar{\alpha}(k(e_2 - e_3)) = -\bar{e}_2' = k'(e_2' - e_3')$ by 7.3.4(i)). The intersection of $k'e_1' + k'e_2'$ and $k'(e_1' + e_3') + k'(e_2' + e_3')$ is $k'(e_1' + e_2')$ whence $\bar{\alpha}(k(e_1 + e_2)) = k'(e_1' + e_2')$. Now we can apply our reasoning to $\tilde{V}_1$ (without changing basis). As result we construct $M_1, \tilde{M}_1, R_1, \bar{\beta}_1, \varphi_1$. We have $R \cap R_1 \supset \{e_1 p + pae_2 + e_3 \mid a \in A\}$ and 7.3.4(i) is applicable on $R \cap R_1$ to both φ and φ_1. So $\bar{\beta}(e_1 + ae_2 + e_3) = e_1' + \varphi(a)e_2' + e_3' = \bar{\beta}_1(e_1 + ae_2 + e_3) = e_1' + \varphi_1(a)e_2' + e_3'$, whence $\varphi = \varphi_1$ and therefore $\beta = \bar{\beta}$ (we have $M = M_1$ by 7.3.5). Since the same argument applies to $\tilde{V}_2$ we have our Lemma.

7.3.6.1. *Remarks.* (i) There is nothing surprising that M turned out to be free. Indeed, if M were "general" module then M would have a form $M = Je_1 + Ae_2 - Ae_3$ with a fractional ideal J. The condition $\dim M/IM = 3$ which we have forces J to be principal.

(ii) Actually we proved more. Namely, we have shown that for any choice of basis e_1, e_2, e_3 in V such that conditions of 3.1 are satisfied (i.e., $\{\bar{\alpha}(ke_i)\}$ is a basis of V' and $\bar{\alpha}(k(e_3 + e_i)) = k'(e_3' + e_i')$) we obtain that free module ΣAe_i satisfies the conclusions of the Theorem.

7.3.7. Lemma. *If we change our choices of bases e_1', e_2', e_3' of V' and e_1, e_2, e_3, of V (subject to conditions stated in* 7.3.1) *then φ, M, $\bar{\beta}$ are replaced by $\varphi_{\text{new}} = \varphi_3$ $M_{\text{new}} = aM$, $a \in k$, $\bar{\beta}_{\text{new}}(ax) = b\bar{\beta}(x)$, $x \in M$, $b \in k'^*$.*

Proof. Let us consider first those changes of e_1, e_2, e_3 such that e_1', e_2', e_3' remain the same. Such changes can be decomposed into a product of a diagonal transformation and of a transformation $D \in GL(M) = GL(3, A)$ with $D \equiv Id \pmod{I}$. For diagonal part we have e_1, e_2, e_3 are replaced by ae_1, be_2, ce_3, $a, b, c \in k$. The condition $\bar{\alpha}(k(e_3 - e_i)) = k'(e_3' + e_i')$ implies $a = b = c$. So M is replaced by aM. Since the definition of φ depends only on ratios of coordinates of a vector, we see that φ is unchanged. The statement about $\bar{\beta}$ is clear.

For $D \in GL(M)$ with $D \equiv 1 \bmod I$ there is nothing to prove, since D does not affect M, φ or $\bar{\beta}$.

Let us now change a basis of V'. Then there exists $g' \in GL(V')$ such that the new basis is $g'e_1', g'e_2', g'e_3'$. If we were able to find $g \in GL(M)$ such that $\varphi^0(g) = g'$ then we would be through. But $GL(M)$ contains all permutation matrices, all unipotent matrices with elements from A and also all dilations $e_1 \to ae_1$, $e_i \to e_i$, $i \geqslant 2$, $a \in A^*$. Since $\varphi\colon A \to k$ and $\varphi^*\colon A^* \to k^*$ are onto it follows that $\varphi^0\colon GL(M) \to GL(V')$ is also surjective, as required.

7.4. To be consistent with our later approach let us restate our theorm.

Theorem. *Let k, k' be fields and let $\bar{\alpha}\colon \mathbf{P}^2(k) \to \mathbf{P}^2(k')$ be a homomorphism of projective planes such that $\alpha(\mathbf{P}^2(k))$ is not contained in a line. Then*

(i) *there exists a unique place $\varphi\colon k \to k'$; let A be its valuation ring,*

(ii) *there exists a unique structure of a projective plane over A on $\mathbf{P}_k^2$,*

(iii) *there exists a unique structure of a plane over $\varphi(A)$ on $\mathbf{P}_{k'}^2$,*

(iv) *there exists a unique $\varphi(A)$-isomorphism $\bar{\beta}\colon {}^{\varphi}\mathbf{P}_A^2 \to \mathbf{P}_{k'}^2$.*

such that

$$\bar{\alpha} = \bar{\beta} \circ \varphi^0.$$

Proof. The only new ingredient (compared with our original Theorem 7.1) is (iii). To prove (iii) we remark (as we did before) that $\tilde{\alpha}(\mathbf{P}^2(k))$ is a subplane of $\mathbf{P}^2(k')$, i.e., $\tilde{\alpha}(\mathbf{P}^2(k)) = \mathbf{P}^2(k'')$ for $k'' \subseteq k'$. This $\mathbf{P}^2(k'')$ gives $\mathbf{P}^2_{k'}$ the k''-structure. It remains to notice that $k'' = \varphi(A)$.

7.4.1. *Remark.* Using language proposed by Tits in [15] we can say that $\tilde{\alpha}$ is a composition of reduction modulo I and of a semi-algebraic isomorphism.

8. APPENDIX 2. Forms of Chevalley Group Schemes

Our aim here is to give a convenient (for our pruposes) way to describe semi-simple group schemes which are split by some étale extension. In the case of fields our description coincides with that of Satake ([13], 13, Remark 1).

8.1. Let $S = \operatorname{Spec} A$, where A is a local domain. Since A is local an étale cover is the same as a Galois extension of A. Let G be a semi-simple group scheme over S and let T be a maximal S-torus of G. We know ([5], X, Corollaire 4.5) that T is split over some Galois extension $S_1 = \operatorname{Spec} A_1$ of S. Let Γ be the Galois group of A_1 over A. Fix some épinglage of G over S_1 with respect to T. Let $\Sigma = \Sigma(G, T)$ be the root system of G with respect to T. One can consider Σ as a constant scheme over S_1 (cf. [5], XIX, Proposition 3.8). Let e_a, $a \in \Sigma$, be the basis over A_1 of the complement to Lie T in Lie G corresponding to the chosen épinglage. Now Γ acts on (Lie G)(A_1) by A-automorphisms and preserves T. Therefore $e_a^{\sigma} \in A_1 e_{\varphi(a,\sigma)}$ for any $a \in \Sigma$ and any $\sigma \in \Gamma$. It is easy to see that $\varphi: \Gamma \otimes \Sigma \to \Sigma$ is an action of Γ on Σ, i.e., it determines a homomorphism $\omega: \Gamma \to \operatorname{Aut} \Sigma$ (Recall, that in our situation we can consider Aut Σ as a constant group scheme over A_1, cf. ([5], XXIV, Proposition 2.6). We shall write σa instead of $\omega(\sigma)a$ for $a \in \Sigma$, $\sigma \in \Gamma$.

Since the e_a are canonically part of a basis of Lie G over A_1, we have $e_a^{\sigma} = d_{a,\sigma} e_{\sigma a}$ with $d_{a,\sigma} \in A_1^*$. Thus we obtain a map $d: \Sigma \otimes \Gamma \to A_1^*$.

We can assume that our Chevally basis of Lie G over A_1 is obtained by a base change from a Chevalley basis of a Lie algebra $\mathfrak{g}_{\mathbf{Z}}$ over $\mathbf{Z}$. Let $N_{a,b}$ be the structure constants of $\mathfrak{g}_{\mathbf{Z}}$: $[e_a, e_b] = N_{a,b} e_{a+b}$ where $N_{a,b} = 0$ iff $a - b \notin \Sigma$. It follows from properties of Chevalley bases that $N_{wa,wb} = \pm N_{a,b}$ for any $w \in \operatorname{Aut} \Sigma$. Define a map ϵ from a subset of $\Sigma \times \Sigma \times \operatorname{Aut} \Sigma$ into $\mathbf{Z}^* = \{\pm 1\}$ by $N_{wa,wb} = \epsilon(a, b, w) N_{a,b}$ if $a + b \in \Sigma$. If $a + b \notin \Sigma$ we do not define ϵ.

8.2. Lemma. *The numbers $d_{a,\sigma} \in A_1^*$ $(a \in \Sigma, \sigma \in \Gamma)$ satisfy relations*

(i) $d_{a,\sigma\tau} = d_{\tau a,\sigma} d_{a,\tau}^{\sigma}$

(ii) $d_{-a,\sigma} = d_{a,\sigma}^{-1}$

(iii) $d_{a+b,\sigma} = \epsilon(a, b, \omega(\sigma))\, d_{a,\sigma} d_{b,\sigma}$ *if* $a + b \in \Sigma$.

Proof. We can assume that G is simply connected. First we have $d_{a,\sigma\tau}e_{\sigma\tau a} = e_a^{\sigma\tau} = (e_a^{\tau})^{\sigma} = (d_{a,\tau}e_{\tau a})^{\sigma} = d_{\tau a,\sigma}d_{a,\tau}^{\sigma}e_{\sigma\tau a}$ whence (i). To prove (ii) consider $[e_a, e_{-a}] = h_a$ (which holds for all $a \in \Sigma$ because G is simply connected). We have $h_a^{\sigma} = h_{\sigma a}$ (by the definition of action of Γ on Σ) and $[e_a, e_{-a}]^{\sigma} = [e_a^{\sigma}, e_{-a}^{\sigma}] = [d_{a,\sigma}e_{\sigma a}, d_{-a,\sigma}e_{-\sigma a}]$ whence (ii).

To prove (iii) we proceed as follows. We have $[e_a, e_b] = N_{a,b}e_{a+b}$ (if $a+b \in \Sigma$). Therefore $d_{a,\sigma}d_{b,\sigma}N_{\sigma a,\sigma b}e_{\sigma(a-b)} = [e_a, e_b]^{\sigma} = (N_{a,b}e_{a+b})^{\sigma} = N_{a,b}d_{a+b,\sigma}e_{\sigma(a+b)}$ whence

$$d_{a,\sigma}d_{b,\sigma}N_{\sigma a,\sigma b} = N_{a,b}d_{a+b,\sigma}\,.$$

This implies (iii) unless $N_{a,b} = 0$ (in A_1). Of course, this happens if and only if $N_{\sigma a,\sigma b} = 0$. But $N_{a,b}$ can be zero in A_1 for $a + b \in \Sigma$ only in few cases. We consider them separately.

(a) $2A = 0$; Σ of type B_2; a, b are short roots with angle $\pi/2$. But then we have $N_{a+b,-b} \neq 0$ whence $d_{a+b,\sigma}d_{-b,\sigma} = d_{a,\sigma}$. Apply (ii) and get $d_{a+b,\sigma} = d_{b,\sigma}d_{a,\sigma}$ as desired.

(b) $2A = 0$; Σ of type G_2; a, b short with angle $2\pi/3$. But then $N_{a+b,-b} \neq 0$ and we proceed as in (a).

(c) $3A = 0$, Σ of type G_2; a, b short with angle $\pi/3$. Then we take in ([14, Sect. 10, No. 2 about G_2] $a = \alpha + \beta$, $b = 2\alpha + 3\beta$. We see that $N_{a,b} = 3N_{-a,a+b}$ whence $\epsilon(a, b, w) = \epsilon(-a, a+b, w)$ for all $w \in \operatorname{Aut}\Sigma$. On the other hand $N_{-a,a+b} = 1$ whence $d_{-a,\sigma}d_{a+b,\sigma} = \epsilon(-a, a+b, \omega(\sigma))d_{b,\sigma}$. Using (ii) and the above remark about ϵ we obtain (iii). This concludes the proof.

8.2.1. *Remarks.* (i) One can work directly in the group and use commutation relations there. In fact, estetically this approach would be better.

(ii) The numbers $d_{a,\sigma}$ depend on a choice of a Chevalley basis.

8.3. Definition. *Given a homomorphism* $\omega: \Gamma \to \operatorname{Aut}\Sigma$ *we call two maps* $d, d': \Sigma \otimes \Gamma \to A_1^*$ *equivalent if there exists a map* $m: \Sigma \to A_1^*$ *satisfying* $m_a m_b = m_{a+b}$ *if* $a, b, a+b \in \Sigma$ *and such that* $d'_{a,\sigma} = m_{\sigma a}^{-1}d_{a,\sigma}m_a^{\sigma}$.

8.4. Now we are going to show that our map $d: \Sigma \otimes \Gamma \to A_1^*$ actually determines a cocycle from Γ with values in the automorphism scheme of a Chevalley group scheme. This implies that d satisfying conditions (i)–(iii) of Lemma 8.2 determines some form of a Chevalley group scheme.

Let G_0 be a Chevalley group scheme over A which is isomorphic to G over A_1 and let T_0 be a maximal split A-torus of G_0. Suppose that we are given an épinglage of G over A_1 and an epinglage of G_0 over A which are obtained from the same epinglage over $\mathbf{Z}$ by base extension. Then there exists an A_1-isomorphism $h: G_0 \to G$ which maps T_0 into T and one epinglage into another. Consider $h^{-1}h^{\sigma} \in \operatorname{Aut}_{S\text{-gr}}G_0$. Then $h^{-1}h^{\sigma}$ is a cocycle from Γ to $(\operatorname{Aut}_{S\text{-gr}}G_0)(A_1)$ but actually its values are in the normalizer N of T_0 in $(\operatorname{Aut}_{S\text{-gr}}G_0)(A_1)$. We have an exact sequence $1 \to \bar{T}_0 \to N \to \mathrm{Aut}\,\Sigma \to 1$ where $\bar{T}_0$ is the centralizer of T_0 in $\mathrm{Aut}_{S-\mathrm{gr}} G_0$. This gives us a map $\delta\colon H^1_{\text{ét}}(\Gamma, N(K)) \to H^1_{\text{ét}}(\Gamma, \mathrm{Aut}\,\Sigma)$. Since G_0 is split we have that $\mathrm{Aut}\,\Sigma$ is a constant group S-scheme (cf. [5, Chap. 24, Proposition 2.6]. Therefore $H^1_{\text{ét}}(\Gamma, \mathrm{Aut}\,\Sigma) = \mathrm{Hom}_{\mathrm{gr}}(\Gamma, \mathrm{Aut}\,\Sigma)$. Thus to our cocycle $h^{-1}h^\sigma$ there corresponds a homomorphism $\omega\colon \Gamma \to \mathrm{Aut}\,\Sigma$. Let us consider the action of $h^{-1}h^\sigma$ on the Chevalley basis of Lie G_0, corresponding to our epinglage. It is easy to see that we have $h^{-1}h^\sigma e_a = d_{a,\sigma} e_{\omega(\sigma)a}$ for $a \in \Sigma$, $\sigma \in \Gamma$ where $d_{a,\sigma}$ are constructed from the given epinglage of G as described in the beginning of this Appendix. To sum up our conclusions we have

8.4. LEMMA. *The data: homomorphism* $\omega\colon \Gamma \to N(A_1)/\bar{T}_0(A_1) \subseteq \mathrm{Aut}\,\Sigma$ *together with a map* $d\colon \Sigma \otimes \Gamma \to A_1^*$ *satisfying* (i), (ii), (iii) *of Lemma* 8.2 *describes a cocycle from* Γ *to* $N_0(A_1)$. *If its image in* $H^1(\Gamma, (\mathrm{Aut}_{S-\mathrm{gr}} G_0)(A_1))$ *determines a group scheme G then* ω *and d are reconstructible from G as described in the beginning of this Appendix.*

8.5. COROLLARY. *The data:* $\omega\colon \Gamma \to N(A_1)/\bar{T}_0(A_1) \subseteq \mathrm{Aut}\,\Sigma$ *together with a map* $d\colon \Sigma \otimes \Gamma \to A_1^*$ *satisfying* (i), (ii), (iii) *of Lemma* 8.2 *describes a semi-simple group scheme over A split by* A_1.

8.5.1. *Remarks.* (i) We actually need only Corollary. Cocycles were introduced because there is a reference [5, Chap. 24, Corollaire 1.18] which says that cocycles do define a form of a Chevalley group scheme.

(ii) We can not take arbitrary $\omega\colon \Gamma \to \mathrm{Aut}\,\Sigma$ because the image of Γ must act on the center of G (or, the same, on the lattice of weights of T). For absolutely simple groups this is a restriction only for groups of type D_{2n}.

Let us make several other observations

8.6. LEMMA. *Let* $d, d'\colon \Sigma \otimes \Gamma \to A_1^*$ *be two maps satisfying* (i), (ii), (iii) *of Lemma* 8.2 *with the same* $\omega\colon \Gamma \to \mathrm{Aut}\,\Sigma$. *If* d, d' *are equivalent in the sense of Definition* 8.3 *the corresponding cocycles are cohomologous with respect to* $\bar{T}_0 = Z_{\mathrm{Aut}\,G_0}(T_0)$.

Proof is evident (if needed at all).

8.6.1. *Remark.* It is easy to see that multiplication on the left by a cocycle from Γ to $\bar{T}_0(A_1)$ maps a cocycle from Γ to $N_0(A_1)$ to a cocycle from Γ to $N_0(A_1)$. The question arises whether it determines an action of $H^1(\Gamma, \bar{T}_0)$ on $H^1(\Gamma, \mathrm{Aut}_{S-\mathrm{gr}} G_0)$.

8.7. LEMMA. *Let* G, G' *be two semi-simple A-group schemes, both having maximal tori* T, T' *split over* A_1. *Suppose that*

(i) *there exists a central A_1-isogeny $\beta: G \to G'$ mapping T to T',*

(ii) *the homomorphisms $\omega, \omega': \Gamma \to \operatorname{Aut} \Sigma$ and the maps $d, d': \Sigma \otimes \Gamma \to A_1^*$ defined by (G, T) and (G', T') respectively are the same with respect to epinglages connected by β.*

Then β is an A-isogeny.

We omit a proof.

8.8. We need only a very special case of above discussions.

THEOREM. *Let A_1 be a quadratic Galois extension of A, $\sigma \in \operatorname{Gal}(A_1/A)$, $\sigma \neq 1$. For every Chevalley group scheme over A there exists its A-form split by A_1 determined by conditions*

(i) $\omega(\sigma) = -1 \in \operatorname{Aut} \Sigma$

(ii) $d_{a,\sigma} = -1$, $d_{a,1} = 1$ *for all* $a \in \Sigma$.

Proof. Since $-1 \in \operatorname{Aut} \Sigma$ preserves any lattice between lattice of weights and lattice of roots, we see that (i) does not lead to a contradiction. It remains to verify (i), (ii), (iii) of Lemma 8.2. The first one, (i), is trivial since $\sigma^2 = 1$, (ii) is also evident. Now (iii) follows from the condition $N_{-a,-b} = -N_{a,b}$ which holds for Chevalley bases.

8.9. COROLLARY. *Let A be an integral domain and suppose that $A((-1)^{1/2})$ is unramified over A. Let $1 \neq \sigma \in \operatorname{Gal}(A((-1)^{1/2})/A)$. Then the data described in Theorem* 8.8 *determines an A-form of a Chevalley group scheme over A. This group scheme is obtained by base change from $\mathbf{Z}$ to A from a form of a Chevalley group scheme over $\mathbf{Z}$ defined similarly (with $A = \mathbf{Z}$).*

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Note added in proof. Donald James has recently generalized and strengthened Theorems 4.3.1 and 7.1 above and also simplified their proofs.

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