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One-Dimensional Affine Group Schemes*

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Our objects of study are affine group schemes, finitely presented and flat, over a domain A. As in [10], we call such group schemes *models* of their generic fibers. In dimension one there are only a few possibilities for the generic fiber, and we are able to obtain a complete classification in some generality:

THEOREM. *Let A be an integrally closed domain with perfect fraction field. Then the smooth group schemes over A with connected affine one-dimensional fibers correspond to rank 2 projective algebras over A.*

The construction of these groups is straightforward; to an algebra B we assign the quotient $R_{B/A}\mathbf{G}_m/\mathbf{G}_m$. The main effort comes in showing that these are the only possibilities. The key to this, and the basic technical idea in the paper, is the use of Néron blow-ups of group schemes over valuation rings. This process has been used before [1, 2] as a tool for resolving singularities, but in fact it also furnishes a good grasp on the structure of models in general. A further instance of this is found in the last section, where we give an analysis of all models (smooth or not) of $\mathbf{G}_m$ over a valuation ring. It seems likely that the technique should also have other applications.

1. NÉRON BLOW-UPS

Let A be a discrete valuation ring with fraction field K, uniformizer π, and residue field k. If G is a flat separated group scheme of finite type over A, and the generic fiber G_K is affine, then G itself is affine [1, p. 30]; throughout this section G will be assumed to be such a model. Our goal here is to acquire information about the blow-up process for passing from G to some

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other model of G_K. Propositions 1.1 and 1.2 are already on record in more generality [1], but we write them out for completeness because they are so simple. The most important result for the later applications is Theorem 1.4 together with Corollary 1.8.

Construction and Universal Property

Let G be an affine model. By flatness we can view the ring of global sections $A[G]$ as an A-subalgebra of $K[G]$. Suppose H is a closed subgroup of the special fiber G_k, defined by some proper ideal J of $A[G]$ containing π. Let G^H be Spec $A[\pi^{-1}J]$, where $A[\pi^{-1}J]$ is taken as a subalgebra of $K[G]$. This G^H is the *Néron blow-up* of H in G.

PROPOSITION 1.1. *G^H is a model of G_K.*

Proof. Since $\pi A[G] \subseteq J$, we have $A[G] \subseteq A[\pi^{-1}J] = A[G^H]$; clearly $K[G^H] = K[G]$. The ideal J is finitely generated, since $A[G]$ is noetherian; and if $J = (\pi, f_1, ..., f_n)$, then $A[G^H] = A[G][\pi^{-1}f_1, ..., \pi^{-1}f_n]$, and in particular G^H is again of finite type. It is flat since by construction $A[G^H]$ is torsion-free. As H is a subgroup, the comultiplication Δ sends J into $J \otimes A[G] + A[G] \otimes J$, so $\Delta(\pi^{-1}J) \subseteq A[G^H] \otimes A[G^H]$; similarly the antipode preserves $A[G^H]$, and the counit ε maps it into A since $\varepsilon(J) \equiv 0 \pmod{\pi}$. ▮

For brevity we call a homomorphism $G' \to G$ a *model map* if it is an isomorphism on the generic fibers.

PROPOSITION 1.2. *The canonical model map $G^H \to G$ sends the special fiber into H, and any model map $G' \to G$ sending the special fiber into H factors uniquely through G^H.*

Proof. Each element of J is in $\pi A[G^H]$ and so goes to zero in $k[G^H] = A[G^H]/\pi A[G^H]$; this proves the first statement. Now take a model map $G' \to G$, and use the isomorphism over K to identify $A[G']$ with a subring of $K[G]$ containing $A[G]$. If G'_k maps into H, then $J \subseteq \pi A[G']$, and so $\pi^{-1}J \subseteq A[G']$. ▮

Remark. The group $G^H(A)$ is canonically identified with those points in $G(A)$ whose reductions modulo π are in H. Thus G^H can be thought of as the group scheme corresponding to a congruence subgroup of G.

Standard Blow-up Sequence

LEMMA 1.3. *A model map that is epimorphic on the special fiber is an isomorphism.*

Proof. Let $G' \to G$ be the map, and again use it to identify $A[G] \subseteq A[G'] \subseteq K[G]$. By hypothesis $A[G]/\pi A[G]$ injects into $A[G']/\pi A[G']$; that is, $A[G] \cap \pi A[G'] = \pi A[G]$. Since the modules are torsion-free with the same K-span, this makes them equal. ∎

THEOREM 1.4. *Any model map is (isomorphic to) a composite of blow-ups.*

Proof. Again we take $A[G] \subseteq A[G']$. If the (closed) image H of G'_k is all of G_k, we have $G' = G$ by the lemma; if not, then $G' \to G$ factors through $G_1 = G^H$. If G'_k does not map onto $(G_1)_k$, we blow up its image to get $G' \to G_2$, and so on. We must show this process eventually stops. But the ideal J defining H is $A[G] \cap \pi A[G']$, so $A[G_1]$ contains $\pi^{-1}J = A[G'] \cap \pi^{-1}A[G]$. Then $A[G_2]$ contains $A[G'] \cap \pi^{-1}A[G_1] \supseteq A[G'] \cap \pi^{-2}A[G]$, and in general $A[G_n]$ contains $A[G'] \cap \pi^{-n}A[G]$. Since $K[G'] = K[G]$, we have $A[G'] \subseteq \bigcup A[G_n]$, and equality occurs at some finite stage because $A[G']$ is finitely generated. ∎

Remarks. (1) We call the sequence G_i constructed in the proof the *standard blow-up sequence* from G to G'. At every stage in it the subgroup blown up maps onto the subgroup blown up at the previous stage, since these subgroups are at each stage just the image of G'_k. Conversely it is easy to see inductively that a sequence of blow-ups with this property is the standard sequence between its beginning and its end.

(2) For blow-ups in general, G^H_k need not map onto H. For example, let $\operatorname{char}(K) = 2$, and let G be the model of $\mathbf{G}_a$ given by $A[x, y]$ where $\Delta x = x \otimes 1 + 1 \otimes x$ and $\pi y = x^2$. Over k we have $x^2 \equiv 0$ and x and y primitive, so $G_k \simeq \mathbf{G}_a \times \boldsymbol{\alpha}_2$. The subgroup $H = \mathbf{G}_a$ is defined by $x \equiv 0$, so $A[G^H]$ is $A[x, y, \pi^{-1}x]$. Here $x = \pi(\pi^{-1}x)$ and $y = \pi(\pi^{-1}x)^2$, so in $k[G] \to k[G^H]$ both x and y go to zero, and thus G^H_k has trivial image in G_k.

(3) The theorem does not imply the ascending chain condition on models between G and G'. For example, let $\operatorname{char}(K) = 2$, let $G = \mathbf{G}_a = \operatorname{Spec} A[x]$, and set

$$A[G_1] = A[x, \pi^{-1}x^2],$$

$$A[G_2] = A[x, \pi^{-1}x^2, \pi^{-3}x^4],$$

$$A[G_3] = A[x, \pi^{-1}x^2, \pi^{-3}x^4, \pi^{-7}x^8],$$

and so on. It is easy to see that the G_i are models, and in fact $G, G_1, G_2, \ldots, G_i$ is the standard blow-up sequence from G to G_i. But all $A[G_i]$ are contained in $A[G^{(e)}] = A[\pi^{-1}x]$.

(4) The group structure has played no essential role in the hypotheses or proofs so far.

Kernel of the Blow-Up

THEOREM 1.5. *Let H be a subgroup of G_k defined by an ideal J in $A[G]$, let I be the augmentation ideal of $A[G]$, and let $r = \dim_k(J/\pi A[G] + IJ)$. Then the kernel of $G_k^H \to G_k$ is isomorphic to a subgroup of $\mathbf{G}_a^r$.*

Proof. Let F be the kernel. The ring $A[G^H]$ is generated over A by $\pi^{-1}J$, so $k[F] = A[G^H]/(\pi, I)A[G^H]$ is generated by the images of the $\pi^{-1}x$ for x in J. Clearly such an image is constant for x in $A[G]$ and zero for x in IJ. Thus $k[F]$ is generated by r elements. We must show now that $\Delta\pi^{-1}x$ is congruent to $\pi^{-1}x \otimes 1 + 1 \otimes \pi^{-1}x$ for each x in $J \cap I$. As H is a subgroup, there is some expression $\Delta x = x \otimes 1 + 1 \otimes x + \sum y_i \otimes z_i$, where in each term either y_i or z_i is in J. The group axiom $(1 \otimes \varepsilon) \circ \Delta = \mathrm{id}$ (existence of identity element) gives us $\sum y_i \otimes \varepsilon(z_i) + 1 \otimes \varepsilon(x) = 0$. As $x \in I$, it follows that $\sum y_i \otimes \varepsilon(z_i) = 0$; and if $z_i \in J$, then $\varepsilon(z_i) \in \pi A \subseteq J$. Thus we can write $\Delta x = x \otimes 1 + 1 \otimes x + \sum y_i \otimes (z_i - \varepsilon(z_i))$. Repeating this on the other side, we may assume $y_i, z_i \in I$. Now $\Delta(\pi^{-1}x) = (\pi^{-1}x) \otimes 1 + 1 \otimes (\pi^{-1}x) + \sum \pi^{-1}(y_i \otimes z_i)$, and the result follows since $\pi^{-1}J \otimes I$ and $I \otimes \pi^{-1}J$ go to zero in $k[F] \otimes k[F]$. ∎

COROLLARY 1.6. *If H contains the connected component of G_k, the kernel is trivial.*

Proof. The ideal $J/\pi A[G]$ in $k[G]$ is idempotent and contained in $I/\pi I$, so $\pi A[G] + IJ = J$. ∎

Remark. The kernel need not be a full vector group, and may even be disconnected. For example, let $\mathrm{char}(K) = 2$, and let $A[G]$ be $A[x, y]$ where x and y are primitive and $\pi y = y^2 + x^4$. (This is a model of a twisted form of $\mathbf{G}_a$.) We get $G^{\{e\}}$ by adjoining $u = \pi^{-1}x$ and $v = \pi^{-1}y$, so $A[G^{\{e\}}] = A[u, v]$; here $v = v^2 + \pi^2 u^4$, and it is easy to see this is a defining equation. Thus over k the kernel, which of course is all of $G_k^{\{e\}}$, is $\mathbb{Z}/2\mathbb{Z} \times \mathbf{G}_a$.

Smooth Connected Case

THEOREM 1.7. *Let G be a smooth model with connected fibers. Let H be a smooth connected subgroup of G_k. Then G^H is smooth with connected fibers, the special fiber maps onto H, and the kernel is a vector group whose dimension is the r in Theorem* (1.5).

Proof. As H is smooth and connected, it has no proper subgroups of the same dimension; thus the image either is H or has lower dimension. In the notation of the last theorem, the kernel either equals $\mathbf{G}_a^r$ or has lower dimension. By flatness [6, $\mathrm{VI_B}$, 4.3] we have $\dim G_k^H = \dim G_K^H = \dim G_K = \dim G_k$, and thus $\dim(\text{kernel}) + \dim(\text{image}) = \dim G_k$. By smoothness,

$\dim G_k = \dim_k(I/\pi I + I^2)$ and $\dim H = \dim_k(I_H/I_H^2)$. If Ω_G^1 denotes the module of differentials on G, then also by smoothness there is a sequence of $k[G]$-modules

$$0 \to J/\pi A[G] + J^2 \to \Omega_G^1/(\pi A[G] + J)\,\Omega_G^1 \to \Omega_H^1 \to 0$$

which is exact and split [5, O_{IV}, §20]. Dividing by I, we get an exact sequence

$$0 \to J/\pi A[G] + IJ \to I/\pi I + I^2 \to I_H/I_H^2 \to 0.$$

Thus $r + \dim H = \dim G_k$. Hence we must have equality of groups in both the kernel and the image. Then G_k^H is smooth and connected since the kernel and image are, and G^H is smooth because it is flat with smooth fibers [5, 17.5.2]. ∎

One can see also how the action of H here on $\mathbf{G}_a^r$ is constructed from the adjoint representation of G_k.

COROLLARY 1.8. *Let $G' \to G$ be a model map of smooth models with connected fibers. Then each stage in the standard blow-up sequence from G to G' is smooth with connected fibers.*

Proof. At each stage the image of G'_k is smooth and connected, since G'_k itself is so. Induction using the theorem shows then that each stage is smooth with connected fibers. ∎

2. Smooth One-Dimensional Groups over Valuation Rings

Over our valuation ring A we are now able to classify all smooth affine G with connected one-dimensional fibers. We will see for instance that $\mathbf{G}_m$ is an *absolutely minimal* model, i.e., $A[\mathbf{G}_m]$ is actually contained in every other Hopf subalgebra spanning $K[\mathbf{G}_m]$; hence we get all models from $\mathbf{G}_m$ by a sequence of blow-ups. Models of other tori can then be treated by descent. For the additive group we have at least that $\mathbf{G}_a$ is a *relatively minimal* model, i.e., every model G has a model map to $\mathbf{G}_a$; thus the same method works, and again from that we get the twisted additive groups.

Models of $\mathbf{G}_a$

LEMMA 2.1. *The additive group $\mathbf{G}_a$ is a relatively minimal model.*

Proof. Let G be any model. Then $K[G] = K[x]$ with x primitive. Changing x by a scalar multiple, we may suppose x is in $A[G]$. Then G has a model map to $\mathbf{G}_a = \operatorname{Spec} A[x]$. ∎

THEOREM 2.2. *Any model of the additive group which is smooth with connected fibers is isomorphic to* $\mathbf{G}_a$.

Proof. Let G be any such model. By the lemma, G maps to $\mathbf{G}_a$. The image of the special fiber is smooth and connected, so is trivial or all of $(\mathbf{G}_a)_k$. In the second case $G \simeq \mathbf{G}_a$ by Lemma 1.3. Otherwise, we blow up the trivial subgroup; this gives us $A[x, \pi^{-1}x] = A[\pi^{-1}x]$, which is just another copy of $\mathbf{G}_a$. Continuing this process, we get $\mathbf{G}_a$ at every stage, and by Theorem 1.4 one of these stages eventually is G. ∎

This result was proved by a quite different method in [10].

Models of $\mathbf{G}_m$

PROPOSITION 2.3. *Let G be a flat affine group over A, and let t be a character of G_K. Then t is in $A[G]$.*

Proof. We may assume $t \neq 1$. Let $M = (K \cdot 1 + K \cdot t) \cap A[G]$, and write $y = t - 1$. Since $\varepsilon(t) = 1$ and $\varepsilon(A[G]) \subseteq A$, we have $M = A \cdot 1 + J \cdot y$ for some A-submodule J of K; J is nonzero, since $A[G]$ spans $K[G]$. For any a in J, the element $\Delta(ay) = ay \otimes 1 + 1 \otimes ay + a(y \otimes y)$ must lie in $M \otimes M$, and hence $a \in J \cdot J$. But clearly $J \subseteq J^2$ forces 1 to be in J. ∎

COROLLARY 2.4. *A diagonalizable group is an absolutely minimal model.*

Proof. The characters span $A[G]$. ∎

THEOREM 2.5. *Let G be smooth with connected fibers, $G_K = \mathbf{G}_m =$ Spec $K[t, t^{-1}]$. Then $A[G]$ is $A[t^{-1}, \pi^{-n}(t-1)]$ for some $n \geqslant 0$.*

Proof. By the corollary, G has a model map to $\mathbf{G}_m$. As in Theorem 2.2, if this is not an isomorphism, the smooth connected image of the special fiber must be the trivial subgroup, which is defined by $t \equiv 1$. Then G maps to $G_1 = \operatorname{Spec} A[t^{-1}, \pi^{-1}(t-1)]$. The special fiber of G_1 is $\mathbf{G}_a$, with coordinate the reduction of $\pi^{-1}(t-1)$. If G_k does not map onto this, the image is again trivial, and G maps to $G_2 = \operatorname{Spec} A[t^{-1}, \pi^{-2}(t-1)]$. This in turn has special fiber $\mathbf{G}_a$, and the construction continues until G is reached. ∎

In Section 4 we will refine this argument to obtain an analysis of all models of $\mathbf{G}_m$. The same argument also applies to models of $\boldsymbol{\mu}_p$: the sequence starts with $\operatorname{Spec} A[t]$ where $t^p = 1$ and yields $\operatorname{Spec} A[\pi^{-n}(t-1)]$ as the other models. Except when char $K = p$, however, this sequence stabilizes; for when $n(p-1) > \operatorname{ord}_A(p)$, the ring is no longer a free A-module, and the special fiber is trivial.

Models of Twisted Tori

THEOREM 2.6. *Let G be smooth with connected fibers, and suppose G_K is a one-dimensional anisotropic torus. Then G has the form* Spec $A[v, w]$ *where $v = w^2 + rvw + sv^2$ and*

$$\Delta v = v \otimes 1 + 1 \otimes v + (r^2 - 2s)\, v \otimes v + 2w \otimes w + rv \otimes w + r\, w \otimes v,$$

$$\Delta w = w \otimes 1 + 1 \otimes w - rs\, v \otimes w - r\, w \otimes w - 2s\, v \otimes w - 2s\, w \otimes v.$$

Proof. There is a separable quadratic extension K' of K with $G_{K'} \simeq \mathbf{G}_m$. Let A' be the integral closure of A in K', so A' is a principal ideal domain (with at most two primes). If A' is a discrete valuation ring, we know from Theorem 2.5 that $A'[G]$ has the form $A'[t^{-1}, b^{-1}(t-1)]$ for some b. But even if there are two primes, the multiples of $t-1$ in $A'[G]$ have bounded denominators, since they do after each localization; hence they equal the multiples of some $b^{-1}(t-1)$. Then here again $A'[t^{-1}, b^{-1}(t-1)]$ is the whole of $A'[G]$, since it is so after localization.

LEMMA 2.7. *The elements $v_m = t^{-[m/2]}(b^{-1}(t-1))^m$ for $m = 0, 1, 2,\ldots$ form an A'-basis of $A'[G]$.*

Proof. When we expand the v_m in powers of t, we see each involves a new power, so they are independent. Clearly they lie in $A'[G]$. One computes that $v_m v_n = v_{m+n}$ unless m and n are both odd, in which case $v_m v_n = v_{m+n} + bv_{m+n+1}$; thus their A'-span is an algebra. It contains $1 = v_0$ and $b^{-1}(t-1) = v_1$ and $t^{-1} = b^2 v_2 - bv_1 + v_0$, so it is all of $A'[G]$. ∎

Let σ be the conjugation of K' over K. By assumption, G_K is not split, so we must have $\sigma(t) = t^{-1}$ in $K[G] \otimes_K K' = K'[t, t^{-1}]$. Furthermore, since A' is faithfully flat over A, the algebra $A[G]$ must be free. In $A'[G] = A[G] \otimes_A A'$, then, the set of σ-fixed elements is $A[G]$. In the basis $\{v_m\}$ we compute that $\sigma(v_m)$ is $(-b/\sigma(b))^m v_m$ for m even and $(-b/\sigma(b))^m (v_m - bv_{m+1})$ for m odd. Since the σ-fixed elements of $A'[G]$ span $A'[G]$, clearly those in each $A'v_{2m-1} + A'v_{2m}$ must span $A'v_{2m-1} + A'v_{2m}$. This tells us first of all that $b/\sigma(b)$ is invertible, but we can go on to deduce a stronger statement. Consider the case $m = 1$. The element $v = (-b/\sigma(b))\, v_2$ is fixed. Another element $cv_1 + av$ then will be fixed iff $c = \sigma(c)[-b/\sigma(b)]$ and $a = \sigma(a) - \sigma(c)\, b$. If this gives us a basis of fixed elements, then c must be invertible in A'. Now our original element b was only determined up to invertible elements, so we may go back and replace b by $\sigma(c)\, b = \sigma(a) - a$.

We now compute fixed elements in $A'[G]$ for b of the form $\sigma(a)-a$. Then $-b/\sigma(b)=1$, so $v=v_2=b^{-2}[t^{-1}-2-t]$ and $w=v_1+av_2=b^{-1}(t-1)+av$ are fixed. Since $v^m=v_{2m}$ and $v^{m-1}w=v_{2m-1}+av_{2m}$, we see that the fixed algebra is precisely $A[v,w]$. From the formulas for v and w it is straightforward to compute Δv, Δw, and the relation; the constants occurring are $r=-(a+\sigma(a))$ and $s=a\sigma(a)$. ∎

Remarks. (1) The proof shows that $\operatorname{Spec} A'[t^{-1}, b^{-1}(t-1)]$ arises by base-extension from A iff b is a unit times an element $\sigma(a)-a$. If, for instance, A' is a valuation ring unramified over A, this puts no restriction on b. But if A' is tamely ramified (char $k\neq 2$), it means that b has odd order. The models over A' arising in this way are here the complement of those arising by extending an A-model of $\mathbf{G}_m$.

(2) Since $b\neq 0$, we have $K'=K(a)$, and thus the last sentence in the proof shows that the quadratic form $\alpha^2+m\alpha\beta+n\beta^2$ is the norm form from K' to K.

(3) The true significance of this group law in v and w will be explained in the next section, where we will extend the result to other base rings.

Models of Twisted Additive Groups

THEOREM 2.8. *Suppose the generic characteristic is p. The smooth affine groups with connected fibers and generic fiber a form of the additive group are the subgroups of* $\mathbf{G}_a^2$ *defined by equations of the form*

$$y^{p^n}=x+a_1x^p+\cdots+a_mx^{p^m}.$$

Proof. We know by [9] that the group over K has such an equation with a_i in K. Replacing y by $\pi^{-s}y$ and x by $\pi^{-sp^n}x$ for large s, we may assume the a_i are in A. Then $G=\operatorname{Spec} A[x,y]$ is one model of the correct form. Let $K'=K(\pi^{p^{-n}}, a_1^{p^{-n}},\ldots, a_m^{p^{-n}})$, and let A' be the integral closure of A in K', so A' is a valuation ring with uniformizer $\lambda=\pi^{p^{-n}}$. We have $A'\otimes A[G]=A'[t]$ with $t^{p^n}=x$.

Suppose now $R\subseteq K[x,y]$ represents another model smooth with connected fibers. By Theorem 2.2 then $A'\otimes R=A'[\lambda^r t]$ for some integer r. Let x_1 be $\pi^r x$, which is in $A'\otimes R$ and hence in R. The primitive elements in $A'\otimes R$ are the A'-span of those in R, since $A\to A'$ is faithfully flat, and hence R contains a primitive y_1 which begins with $\lambda^r t$. Then y_1 has degree p^n over $K'(x_1)=K'(t^{p^n})$, so $K(x_1,y_1)$ must be all of $K(x,y)$. As y_1 is a p-polynomial in t, its equation is $y_1^{p^n}=x_1+b_1x_1^p+\cdots+b_sx_1^{p^s}$ for some b_i in K. Since $A'[\lambda^r t]$ is integral over $A[x_1]$, the b_i are actually in A. The form of the equation shows now that $A[x_1, y_1]$ is integrally closed. But R is integral over it and has the same fraction field. Thus $R = A[x_1, y_1]$. ∎

COROLLARY 2.9. *Let G be smooth with connected one-dimensional fibers, and suppose G_K is a rational affine curve. Then G has the form described in Theorem* 2.6 *for some v and w.*

Proof. The group must be G_a or G_m over the algebraic closure [3, p. 257]. Looking at the equations of twisted forms, one sees that the rational G_K are tori and G_a and the groups $y^2 = x + cx^2$ in characteristic 2. (We are grateful to Russell for pointing this out.) For $y^2 = x + cx^2$, take $v = x$ and $w = y$; for G_a, take $w = x$ and $v = x^2$; for $A[t^{-1}, b^{-1}(t-1)]$, take $w = b^{-1}(t-1)$ and $v = b^2(t^{-1} - 2 + t)$; for anisotropic G_k, use Theorem 2.6. ∎

3. GLOBAL CLASSIFICATION OF SMOOTH ONE-DIMENSIONAL GROUPS

In this section we obtain in fair generality a classification of smooth affine group schemes with connected one-dimensional fibers. First we construct a collection of such groups explicitly. Then for an arbitrary model G we use the results of the previous section to produce one of our collection which agrees with G over all discrete valuation rings. A theorem of Raynaud then shows the two are actually isomorphic.

Construction of the Groups

THEOREM 3.1. *Let A be a ring, and let B be an A-algebra with unit which is a projective rank* 2 *A-module. Let G be the Weil restriction of G_m from B to A. Then the quotient $G_{(B)} = G/\mathbf{G}_m$ exists and is a smooth affine group scheme over A. If B has basis* 1, *h with $h^2 = mh - n$, then $G_{(B)} = \operatorname{Spec} A[v, w]$ where $v = w^2 + mvw + nv^2$ and*

$$\Delta v = v \otimes 1 + 1 \otimes v + (m^2 - 2n)\, v \otimes v + 2w \otimes w + m\, v \otimes w + m\, w \otimes v,$$

$$\Delta w = w \otimes 1 + 1 \otimes w - mn\, v \otimes v - m\, w \otimes w - 2n\, v \otimes w - 2n\, w \otimes v.$$

Proof. The result is Zariski-local on $\operatorname{Spec} A$, so by localizing we may assume that B is free with a basis 1, h. An element $x + yh$ is invertible iff its norm $z = x^2 + mxy + ny^2$ is invertible, and thus $A[G]$ is $A[x, y, z]/(z(x^2 + mxy + ny^2) - 1)$. The functions $v = y^2 z$ and $w = xyz$ are invariant under the subgroup $\mathbf{G}_m$ defined by $y=0$, and it is a simple computation to show that v and w satisfy the relation and have Δv, Δw as stated. (Note for this that z is the reciprocal of the norm and hence is a character, i.e., $\Delta z = z \otimes z$.) We have also $0=\varepsilon(x)=\varepsilon(w)$, and the coinverse sends v to v and w to $-w-mv$.

Suppose first $A=K$ is a field. Then $K[v, w]$ is a proper nontrivial Hopf subalgebra of $K[G]$. Hence Spec $K[v, w]$ is a group scheme; being a proper nontrivial quotient of G, it is smooth and connected of dimension one. The polynomial $w^2+mvw+nv^2-v$ is easily seen to be irreducible over K, so it is the defining relation for the ring.

Now for general A let G' be Spec $A[V, W]/W^2+mVW+nV^2-V$, with Δ defined by the formulas for v and w. The identities needed for this to be a group scheme are satisfied, since for indeterminate m and n they hold over $K=\mathbf{Q}(m, n)$ and hence over $\mathbf{Z}[m, n]$. The ring is flat over A, being actually free of rank 2 over $A[V]$. By the argument over fields, all fibers are connected of dimension one, and G' is smooth over A since W spans the augmentation ideal modulo its square.

Sending V, W to v, w gives a homomorphism $G \to G'$. Clearly $\mathbf{G}_m$ is in the kernel. But if evaluation at some point gives $0=v=w$, then $0=y^2=xy$, since z is invertible. The x, y, z relation then gives $1=zx^2$, so x is invertible, and from $xy=0$ we get $y=0$. Thus the kernel is precisely $\mathbf{G}_m$. Both G and G' are smooth; one observes the map on Lie algebras is surjective, and by translation the same is true on all fibers, so $G \to G'$ is smooth [4, p. 114] and hence flat. Since the fibers of G' are connected, the map is faithfully flat. Thus G' is the (*fppf*) quotient $G/\mathbf{G}_m = G_{(B)}$. ∎

Structure of the Fibers

The passage from B to $G_{(B)}$ obviously commutes with base change in A, and in particular the fibers of $G_{(B)}$ are described by the following result.

PROPOSITION 3.2. *Let $A=K$ be a field. Then $G_{(B)}$ is*

(a) $\mathbf{G}_m$ *if $B=K\times K$,*

(b) *a twisted torus if B is a separable quadratic field extension,*

(c) $\mathbf{G}_a$ *if B has nontrivial nilpotents, and*

(d) *a twisted additive group if B is an inseparable field extension (so* char $K=2$).

Proof. If B is $K\times K=K[h]/h^2-h$, we have $m=1$ and $n=0$; thus by inspection $G_{(B)}=\mathbf{G}_m$. Even more trivially, if B is $K[h]/h^2$, we have $m=n=0$ and $G_{(B)}=\mathbf{G}_a$. We now need a purely computational result:

LEMMA 3.3. *Suppose the basis in B is changed from h to $h' = \alpha h + \beta$, with α invertible. Then*

$$
\begin{aligned}
m' &= \alpha m + 2\beta, \\
n' &= \alpha^2 n + \alpha\beta m + \beta^2, \\
v &= \alpha^2 v', \\
w &= \alpha w' + \alpha\beta v'. \quad \blacksquare
\end{aligned}
$$

Now suppose B is a separable quadratic extension. After base extension to B, the algebra becomes $B \otimes_K B \simeq B \times B$, so $G_{(B)}$ becomes $\mathbf{G}_m$; hence certainly $G_{(B)}$ is a torus over K. If over K we have $G \simeq \mathbf{G}_m \simeq G_{(B')}$ with $B' = K[h']/((h')^2 - h')$, consider the functions v', w' in $K[G]$ corresponding to this. Over the larger field the algebras are isomorphic, and we have a linear change of variable from h to h'. Since it takes v, w to v', w', and all these are defined over K, we see that α and β lie in K; but that is impossible, since B is not isomorphic to the split algebra over K. Thus $G_{(B)}$ is anisotropic. The same argument shows that $G_{(B)}$ is a twisted form of $\mathbf{G}_a$ if B is an inseparable field extension. ∎

The argument here actually proves the following result:

PORISM 3.4. *Suppose $G_{(B)} \simeq G_{(B')}$ over a field. Then $B \simeq B'$, and v, w are related to v', w' as in Lemma* 3.3. ∎

Classification Theorem

THEOREM 3.5. *Let A be an integrally closed domain with fraction field K. Let G be a smooth finitely presented group scheme over A with connected one-dimensional fibers. Assume that G_K is a rational curve. Then G is of the form $G_{(B)}$ for a unique algebra B.*

Proof. Note first that G_k is affine, since the projective line cannot carry a group scheme structure. Also, since G is finitely presented, it comes by base extension from a group scheme over a subring finitely generated over $\mathbb{Z}$; the subring remains finitely generated if we replace it by its integral closure in any finite field extension [5, §8 and §7]. Thus we may assume A is noetherian.

The sections of the structure sheaf of G on the first two infinitesimal neighborhoods of the identity give us A-modules which (whether or not G is affine) we denote by I/I^2 and I/I^3. By smoothness these are projective. We let B be the rank 2 A-module $(I/I^2) \otimes_A (I/I^3)^\vee$, where the superscript $\vee$ denotes the dual module. Clearly formation of this B commutes with localization of A.

Over the field K we can by Corollary 2.9 write G_K in the desired form, say with coordinates v, w and constants m, n. If f and g are dual to the classes of w and v in I_K/I_K^3, then $[w] \otimes f$ and $[w] \otimes g$ are a basis of B_K. We define a multiplication of B_K by letting $[w] \otimes f$ be the unit and making $h = [w] \otimes g$ satisfy $h^2 = mh - n$. Porism 3.4 shows easily that this multiplication is canonical, i.e., independent of the particular v and w chosen.

Now let q be a height one prime of A. Corollary 2.9 shows that our group has the desired form over the discrete valuation ring A_q. If we choose v and w as in Corollary 2.9, the m and n are in A_q, and thus B_q is an A_q-subalgebra of B_K. Now A is integrally closed noetherian, and B projective, so we have $B = \bigcap B_q$ inside B_K. Hence B is an A-subalgebra of B_K. At this point we see that if we had started with $G = G_{(B')}$, we would here recover B' as our B; thus the uniqueness is demonstrated.

In any case we can now form the group $G_{(B)}$ for our algebra B. Suppose first G_K is a torus. Then there is an isomorphism $G_K \simeq (G_{(B)})_K$, unique up to inversion, and by construction it extends to isomorphisms over A_q for all height one primes. Since both G and $G_{(B)}$ are smooth with connected fibers, it follows by a theorem of Raynaud [8, p. 130] that in fact the map extends to an isomorphism over A. In the additive case, the K-isomorphism is unique only up to scalars. But the construction makes the A-modules I/I^2 for the two groups isomorphic, and we take a map which matches them up; the same argument then remains valid. ∎

COROLLARY 3.6. *All such groups G are affine.* ∎

This is special to the one-dimensional case. Indeed, Raynaud [8, p. 116] has given an example where $A = \mathbf{C}[X_1, X_2]$ and G is a smooth non-affine model of $\mathbf{G}_m \times \mathbf{G}_m$ with connected fibers.

COROLLARY 3.7. *Let A be an integrally closed domain with perfect fraction field. Then the smooth affine group schemes over A with connected one-dimensional fibers are in one-to-one correspondence with the rank 2 A-algebras.* ∎

Remarks. (1) It is a pure formality to extend the theorem and its corollaries to non-affine integral normal base schemes.

(2) One can check that when A is a discrete valuation ring, the operation of blowing up the identity in G corresponds to replacing B by $A + \pi B$. In general, if $G_{(B)}$ and $G_{(B')}$ are two models of G_K, the identification of generic fibers extends to a map $G_{(B)} \to G_{(B')}$ iff $B \subseteq B'$.

(3) It is well known [6, X.7.2] that one-dimensional tori over Spec A are classified by rank 2 etale coverings. The theorem shows that more general smooth models are classified by the ramified coverings

Spec $B \to$ Spec A. It would be interesting to find a more geometric process for recovering B from the model.

Models of $\mathbf{G}_a$ *and* $\mathbf{G}_m$

COROLLARY 3.8. *Let A be an integrally closed domain. Smooth models of $\mathbf{G}_a$ with connected fibers are classified by* Pic(A), *the group of invertible fractional ideals modulo scalars. In particular, they are locally isomorphic to $\mathbf{G}_a$ in the Zariski topology.*

Proof. A subalgebra B of $K[h]/h^2$ is spanned by 1 and the multiples of h that it contains. These form an invertible fractional ideal, and an algebra isomorphism will change the ideal by a scalar multiplication. Explicitly, we obtain that $A[G]$ is the symmetric algebra on I/I^2. ∎

Under more special hypotheses, this result was proved in [10].

COROLLARY 3.9. *Let A be an integrally closed domain. Smooth models of $\mathbf{G}_m$ with connected fibers correspond to invertible ideals in A.*

Proof. Here a subalgebra B of $K[h]/h^2 - h$ must have the form $A \oplus Jh$ for some invertible ideal J contained in A. If t is the basic character of $\mathbf{G}_m$, we find explicitly $A[G] = A[t^{-1}, J^{-1}(t-1)]$. ∎

Remark. The G in this corollary is a group scheme version of a congruence subgroup: $G(A) = \{\alpha \in \mathbf{G}_m(A) \mid \alpha \equiv 1 \pmod{J}\}$. Locally (and globally if $\mathrm{Pic}(A) = 1$) we have J principal, and $A[G]$ has the form $A[x, y]/(x(by+1) - 1)$.

Lifting to Characteristic Zero

COROLLARY 3.10. *Let k be a field of positive characteristic. Let G_k be a smooth connected affine group over k of dimension one. The following are equivalent:*

(1) *G_k lifts to a smooth affine group over any valuation ring of mixed characteristic with residue field k.*

(2) *$G_k \simeq G_{(B)}$ for some k-algebra B.*

(3) *G_k is a rational curve.*

Proof. We know by Proposition 3.2 and Corollary 2.9 that (2) and (3) are equivalent. One can lift $G_{(B)}$ just by lifting the m and n in the quadratic polynomial defining B, so (2) implies (1). Suppose conversely that G lifts G_k. Replacing G by its connected component [6, $\mathrm{VI_B}$.3.10], we may assume

G_K is connected. Then the theorem shows G has the form $G_{(B)}$, and the same is true of its reduction. ∎

Observe that a group $G_k \simeq G_{(B)}$ actually lifts to any ring that maps onto k.

4. Arbitrary Models of $\mathbf{G}_m$ over Valuation Rings

We now return to the setting and notation of the first two sections and use the blow-up method to analyze *all* models of $\mathbf{G}_m$ over the discrete valuation ring A. The same procedure can be applied to models of $\mathbf{G}_a$, but they have already been studied in [10] by a different method.

Reduced Polynomials

We first need an algebraic result for which we can let A be any integral domain with fraction field K. Let $u_0, u_1, ..., u_m$ be indeterminates, and let $r(0), r(1), ..., r(m-1)$ be integers greater than 1. Call a monomial *reduced* if each u_i for $i < m$ occurs in it with exponent less than $r(i)$, and call a polynomial reduced if it involves only reduced monomials. Let B be the quotient of $A[u_0, ..., u_m]$ by relations of the form

$$\begin{aligned} a_1 u_1 &= u_0^{r(0)} + P_0(u_0), \\ a_2 u_2 &= u_1^{r(1)} + P_1(u_0, u_1), \\ &\vdots \\ a_m u_m &= u_{m-1}^{r(m-1)} + P_{m-1}(u_0, u_1, ..., u_{m-1}), \end{aligned} \tag{*}$$

where the a_i are nonzero elements of A and the P_i are reduced polynomials.

Assign to u_{i+1} the weight $r(0)\, r(1) \cdots r(i)$, so that any reduced polynomial in $u_0, ..., u_i$ has weight less than that of u_{i+1}. By induction one shows now that every polynomial has the same image in B as a reduced polynomial. Indeed, it is enough to prove this for monomials. If $\prod u_i^{s(i)}$ has say $s(j) \geqslant r(j)$, replace it by

$$\left(\prod_{i \neq j} u_i^{s(i)} \right) u_j^{s(j)-r(j)} [a_{j+1} u_{j+1} - P_j(u_0, ..., u_j)].$$

All terms but the first have smaller weight, and the first term has the same weight and smaller total degree.

In the map to $B \otimes_A K \simeq K[u_0]$, the image of u_i has by induction u_0-degree equal to the weight of u_i. Hence the reduced monomials all have different degrees and are independent. Thus B is a free A-module with basis the reduced monomials. Its localization $B[u_0^{-1}]$ then is flat and injects into $B[u_0^{-1}] \otimes_A K = K[u_0, u_0^{-1}]$. We restate this result in the way it will be used:

PROPOSITION 4.1. *Suppose an A-subalgebra of* $K[u_0, u_0^{-1}]$ *is generated by* u_0, u_0^{-1}, *and elements* $u_1, \ldots, u_m$ *satisfying equations* (*). *Then these generators and relations define the algebra.*

Actually the reduced polynomials are a basis of B even when some a_i are zero, or indeed for any a_i in any ring A; this follows by base change from the result over $\mathbb{Z}$ with indeterminates adjoined. In essence this result is contained in [10].

Residue Characteristic Zero

We turn now to a model G of $\mathbf{G}_m$ over the valuation ring A. By Proposition 2.3 we know that G is represented by a subalgebra $A[G]$ of $K[\mathbf{G}_m] = K[t, t^{-1}]$ containing t and t^{-1}. The structure of G is particularly simple when $\mathrm{char}(k) = 0$, and we treat that case first.

THEOREM 4.2. *Suppose* $\mathrm{char}(k) = 0$. *Let* G *be a model of* $\mathbf{G}_m$ *over* A, *and assume* $G \neq \mathbf{G}_m$. *Then* $A[G] = A[t^{-1}, t, y]/(\pi^d y - t^n + 1)$ *for some* $d, n \geqslant 1$.

Proof. By the general theory we know G is reached from $\mathbf{G}_m$ by a standard blow-up sequence, and we merely have to verify inductively that each stage has this form. The only proper k-subgroups of $\mathbf{G}_m$ are the $\boldsymbol{\mu}_n$ for various $n \geqslant 1$, defined by $t^n - 1 \equiv 0$. Thus the first blow-up is $A[t^{-1}, t, z_1]$ with $\pi z_1 = t^n - 1$. One has $\Delta z_1 = z_1 \otimes 1 + 1 \otimes z_1 + \pi z_1 \otimes z_1$, and thus over k the first blow-up group reduces to $\boldsymbol{\mu}_n \times \mathbf{G}_a$. The image H of G_k in this reduction must map onto the image $\boldsymbol{\mu}_n$ at the previous stage, and therefore $H = \boldsymbol{\mu}_n$ or $H = \boldsymbol{\mu}_n \times \mathbf{G}_a$, since $\mathbf{G}_a$ has no proper subgroups in characteristic zero. If we do not yet have G at this stage, the subgroup is proper and hence equals $\boldsymbol{\mu}_n$, defined by $z_1 \equiv 0$. Thus we adjoin $z_2 = \pi^{-1} z_1$, getting $A[t^{-1}, t, z_2]$ with $\pi^2 z_2 = t^n - 1$. Again this reduces to $\boldsymbol{\mu}_n \times \mathbf{G}_a$, and the same argument applies here and at every subsequent stage. ∎

The "Néron model" of [7] does not appear here because it is not of finite type.

Residue Characteristic p: The Statement

The analysis when $\mathrm{char}(k) = p$ is similar in outline but more complicated because $\mathbf{G}_a$ has non-trivial subgroups. We first state the theorem, give then a lemma and the induction step, and conclude with further comments and corollaries.

THEOREM 4.3. *Suppose* $\mathrm{char}(k) = p$. *Let* G *be a model of* $\mathbf{G}_m$ *over* A, *and assume $G \neq \mathbf{G}_m$. Then $A[G]$ can be written as $A[t^{-1}, t, y_1, \dots, y_m]$ with relations of the form*

$$\pi^{d(1)} y_1 = t^n - 1,$$
$$\pi^{d(2)} y_2 = y_1^{r(1)} + P_1(y_1),$$
$$\vdots$$
$$\pi^{d(m)} y_m = y_{m-1}^{r(m-1)} + P_{m-1}(y_1, \dots, y_{m-1})$$

for which the following statement are true:

(1) *The $d(i)$ and n are $\geqslant 1$, the $r(i)$ are non-trivial powers of p, and the P_i are polynomials without constant term which are reduced with respect to the $r(j)$.*

(2) *P_i is congruent modulo π to a p-polynomial in y_i plus a reduced polynomial in $y_1, \dots, y_{i-1}$.*

(3) *Δy_i is congruent modulo π to $y_i \otimes 1 + 1 \otimes y_i$ plus a reduced expression in $y_1 \otimes 1$, $1 \otimes y_1, \dots, y_{i-1} \otimes 1$, $1 \otimes y_{i-1}$ with no constant term.*

(4) *The schemes* Spec $A[t^{-1}, y_1, \dots, y_i]$ *are group schemes occurring as some of the stages in the standard blow-up sequence from $\mathbf{G}_m$ to G.*

(5) *The special fiber G_k is $\boldsymbol{\mu}_n \times U$ for some unipotent U that contains a subgroup $\mathbf{G}_a$ with $U/\mathbf{G}_a$ finite.*

Remarks. (1) By Proposition 4.1, if the y_i satsify these relations, they are defining relations; in particular, the special fiber is found by reading the equations modulo π. Observe also that G is a complete intersection.

(2) Statement (5) is included only for clarity; it is a consequence of the more detailed statements preceding it.

(3) We may assume that P_i includes no terms divisible by $\pi^{d(i+1)}$.

Homomorphisms to $\mathbf{G}_a$

LEMMA 4.4. *Let G be as in the conclusion of Theorem* 4.3. *Let φ: $G_k \to \mathbf{G}_a$ be a homomorphism. Then φ has the form $\sum c_s y_m^{p^s} + h(y_1, \dots, y_{m-1})$ for some reduced polynomial h with no constant term.*

Proof. The homomorphism φ is of course identified with an element in $k[G] = k[t, y_1, \dots, y_m]$. Since φ must be trivial on $\boldsymbol{\mu}_n$, it is constant on $\boldsymbol{\mu}_n$-cosets; hence t does not occur when we write φ in its unique reduced form. Let ψ be the restriction of φ to the subgroup $\mathbf{G}_a$ of U given by $y_1 = \cdots = y_{m-1} = 0$. Since ψ is additive, it is given by some p-polynomial $\sum c_s y_m^{p^s}$. Let α: $U \to \mathbf{G}_a$ be the map which replaces y_i-coordinates by zero for $i < m$; by condition (3) in the theorem this is a $\mathbf{G}_a$-space map, i.e., $\alpha(u + g) = \alpha(u) + g$

for u in U and g in $\mathbf{G}_a$. Hence $\varphi - \psi\alpha$ is constant on $\mathbf{G}_a$-cosets and passes to $U/\mathbf{G}_a$, so it is given by a function $h(y_1,..., y_{m-1})$. There is no constant term because φ vanishes on the identity element where all y_i are zero. ▮

The Induction

An arbitrary model G' can be built up by its standard blow-up sequence from $\mathbf{G}_m$. The first blow-up, together possibly with others immediately following, has the same form as when $\mathrm{char}(k) = 0$ and gives y_1 in the theorem. Inductively we suppose we have some G satisfying the theorem as one stage in the blow-up sequence; if H is the image of G'_k in G_k, we must show G^H satisfies the theorem. We may of course assume $H \neq G_k$.

First we show that H is the kernel of a homomorphism $\varphi: G_k \to \mathbf{G}_a$. For this, let G'' be the stage in the blow-up sequence with $A[G''] = A[t^{-1}, y_1,..., y_{m-1}]$. The kernel V of $G_k \to G''_k$ is a copy of $\mathbf{G}_a$ with coordinate y_m. The subgroup H must map onto the image of G'_k in G''_k. But by the construction of blow-ups, all of G_k maps into this image. Thus VH is all of G_k, and $G_k/H \simeq V/H \cap V$. This quotient of $V \simeq \mathbf{G}_a$ is non-zero, since $H \neq G_k$, and so it is isomorphic to $\mathbf{G}_a$.

The homomorphism φ has the form established in the previous lemma. It must involve y_m, since $V/H \cap V$ is non-trivial. Suppose first it is linear in y_m, so H is defined by an equation $0 = \varphi = y_m + h(y_1,..., y_{m-1})$. The blow-up then is given by adjoining $y'_m = \pi^{-1}[y_m - h'(y_1,..., y_{m-1})]$, where h' is a lifting of h. We can then drop y_m as a generator, replacing it by y'_m. The equation involving y_m is replaced by

$$\pi^{d(m)+1} y'_m = y_{m-1}^{r(m-1)} + P_{m-1}(y_1,..., y_{m-1}) + \pi^{d(m)} h'(y_1,..., y_{m-1}).$$

The statements in the theorem all trivially remain true.

Now suppose that φ is of higher degree in y_m. Then the blow-up has a new generator satisfying

$$\pi y_{m+1} = y_m^{r(m)} + P_m(y_1,..., y_m),$$

where the right-hand side is a lifting of φ (which we may assume is again reduced with no constant term). The statements in the theorem follow from the lemma; in particular, (3) follows from $\Delta\varphi = \varphi \otimes 1 + 1 \otimes \varphi$. Thus Theorem 4.3 is proved. ▮

Corollaries and Remarks

COROLLARY 4.5. *For G as in the theorem, there is an exact sequence $1 \to \boldsymbol{\mu}_n \to G \to G' \to 1$ where G' is a model of $\mathbf{G}_m$ with unipotent special fiber.*

Proof. Take G', with variables t' and y'_i, to have the same equations as G except that n is set $= 1$. We map $G \to G'$ by $t' = t^n$ and $y'_i = y_i$. Looking at reduced polynomials, we see that $A[t, y_1, \ldots, y_m]$ is free over $A[t', y'_1, \ldots, y'_m]$ with basis $1, t, \ldots, t^{n-1}$; the same then remains true after adjoining $(t')^{-1}$. In particular $G \to G'$ is faithfully flat. The kernel is obviously $\boldsymbol{\mu}_n$; explicitly, the map to $A[X]/X^n - 1$ sends t to X and t^{-1} to X^{n-1} and y_i to 0. ∎

COROLLARY 4.6. (1) *The special fiber is smooth iff n is prime to p and each P_i contains a term $b_i y_i$ with b_i invertible.*

(2) *The special fiber is connected iff n is a power of p and the y_i-terms in P_i all have coefficients divisible by π.*

Proof. The condition on n is of course the obvious restriction on $\boldsymbol{\mu}_n$; the rest refers to U. But U has a subgroup $\mathbf{G}_a$ with $U/\mathbf{G}_a$ finite, so U is smooth or connected iff $U/\mathbf{G}_a$ is. The finite group scheme $U/\mathbf{G}_a$ has a filtration with successive quotients represented by $k[y_i]/(y_i^{r(i)} + y_i\text{-terms in } P_i (\mathrm{mod}\ \pi))$, so it is smooth or connected iff each of these is so. Thus the condition reduces to the standard one for subgroups of $\mathbf{G}_a$. ∎

The two requirements are incompatible if $m > 1$ or $n > 1$, so we recover the classification Theorem 2.5 of smooth models with connected fibers.

Remarks. (1) Induction shows that the number of steps in the standard blow-up sequence from $\mathbf{G}_m$ to G is precisely $d(1) + d(2) + \cdots + d(m)$. If $d(m) = 1$, the stage preceding G is that where the generator y_m and its relation are omitted. If $d(m) > 1$, the preceding stage has $d(m)$ decreased by 1 and terms divisible by $\pi^{d(m)-1}$ dropped from P_{m-1}.

(2) As we observed at the start of the section, the polynomial y_i in $K[t]$ has degree $nr(1) \cdots r(i-1)$, larger than the degree of any reduced polynomial in t and the lower y_j. Hence when we write the elements of $K[t] \cap A[G] = A[t, y_1, \ldots, y_m]$ in reduced form, those of degree $< n$ involve only t, those of degree $< nr(1)$ involve only t and y_1, and so on. Thus the generators of $K[t] \cap A[G]$ are built up in the obvious algebraic way: go up by degrees, taking in new elements if they are not in the ring generated by those previously available. Indeed, this starts with t, then gives nothing else until y_1 in degree n, then nothing until y_2 in degree $nr(1)$, and so on. In particular, this shows that the numbers n, $r(i)$, $d(i)$ are unique.

Also we observe that y_i as a polynomial in $K[t]$ has non-zero constant term. For otherwise we would have $t^{-1}y_i$ in $K[t] \cap A[G]$; being of lower degree, it should be in $A[t, y_1, \ldots, y_{i-1}]$, which is impossible since its reduced expression in $K[t, y_1, \ldots, y_{i-1}]$ involves $\pi^{-d(i)}$.

(3) The polynomials P_i satisfy further conditions, but they seem to be too complicated to be worth working out. We should at least mention that

two statements known [10] for models of $\mathbf{G}_a$ are not true here: the P_i need not be p-polynomials in y_i, and the $d(i)$ for $i > 1$ need not be less than $\operatorname{ord}_A(p)$. For example, take $A = \mathbf{Z}_3[3^{1/2}]$, with $\pi = 3^{1/2}$. Let $G = \operatorname{Spec} A[t, y_1, y_2]$, with $\pi y_1 = t - 1$ and $\pi^d y_2 = y_1^3 + \pi y_1^2 + y_1$; for any d this is a model of $\mathbf{G}_m$ (in fact $\Delta y_2 = y_2 \otimes 1 + 1 \otimes y_2 + \pi^{d+3} y_2 \otimes y_2$). What happens here is that the equation $y_1^3 + \pi y_1^2 + y_1 = 0$ actually defines a subgroup which is flat (a model of $\boldsymbol{\mu}_3$) inside $\operatorname{Spec} A[t, y_1]$.

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