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Monomorphisms between Subgroups of Groups of Type G_2

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Abstract monomorphisms of a full subgroup of a group of type G_2 into another group of type G_2 are shown to have a standard decomposition into a field monomorphism and a special isogeny of algebraic groups.

0. INTRODUCTION

We prove here a theorem about monomorphisms of full subgroups of algebraic groups of type G_2. The statement is standard and the proof is not difficult although circumferential.

0.1. If G is a group of type G_2 defined over an infinite field k, we denote by Y the set of singular k-tori of G (i.e., k-tori T such that $Z_G(T)$ is reductive with semi-simple part of type A_1). We say (after O. T. O'Meara and his school) that an (abstract) subgroup H of $G(k)$ is *full* if $|H \cap T| > 4$ for any $T \in Y$.

0.2. MAIN THEOREM. *Let k and G be as above and let k' and G' be another infinite field and a group of type G_2 over k'. Let H be a full subgroup of G and let $\alpha: H \to G'(k')$ be a group monomorphism with $\alpha(H)$ dense in G'. Then there exist a unique field homomorphism $\varphi: k \to k'$ and a unique special k-isogeny $\beta: {}^{\varphi}G \to G'$ such that $\alpha(h) = \beta(\varphi^{\circ}(h))$ for $h \in H$. The special isogeny β is an isomorphism unless* char $k = 3$.

Here we denote (after [3]) by ${}^{\varphi}G$ the algebraic group over $\varphi(k)$ obtained from G by base change φ and by φ° the canonical isomorphism φ°: $G(k) \to {}^{\varphi}G(\varphi(k))$.

0.3. The history of the related results is as follows. R. Steinberg [8] proved the above result for the case $H = G(k)$, α an isomorphism, and G

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split over k. Then A. Borel and J. Tits [3] proved the above result in the case $H = G(k)$ and G is split over k. Needless to say, these authors proved other, much more general results.

0.4. To describe the main steps of our proof we first recall [1, 4, 10] that every k-group of type G_2 can be realized as a group of automorphisms of an octave algebra over k. We denote by $\mathcal{O}$ and $\mathcal{O}'$ the octave algebras corresponding to G and G'. Let $S_i(\mathcal{O})$ be the set of i-dimensional semi-simple subalgebras of $\mathcal{O}$. For $S \in S_i(\mathcal{O})$ denote by G_S the pointwise stabilizer of S in G. Our first (and main) step is to show that the Zariski closure of $\alpha(H \cap G_S)$ is not dense in G' for any $S \in S_2(\mathcal{O})$. It follows then that the Zariski closure of $\alpha(H \cap G_S)$ is an algebraic subgroup of type A_2 of G'. If char $k' \neq 3$, this Zariski closure is of the form $G'_{S'}$ for some $S' \in S_2(\mathcal{O}')$. If char $k' = 3$, then the same property holds for all $S \in S_2(\mathcal{O})$ if it holds for one $S \in S_2(\mathcal{O})$. We consider first the case when the Zariski closure of $\alpha(H \cap G_S)$ is of the form $G'_{S'}$ for $S' \in S_2(\mathcal{O}')$. Then there is a map α_2: $S_2(\mathcal{O}) \to S_2(\mathcal{O}')$ given by: closure of $\alpha(H \cap G_S)$ is $G'_{\alpha_2(S)}$. Moreover, $H \cap G_S$ is full in G_S in the sense of [5, §3.4] and therefore [5, §4] is applicable. Thus to the monomorphism $\alpha: H \cap G_S \to G'_{\alpha_2(S)}$ there correspond a unique ring monomorphism $\varphi_S: S \to \alpha_2(S)$ such that $\varphi_S(k) \subseteq k'$, a unique k'-isomorphism of algebraic groups $\beta_S: {}^{\varphi_S|k}G_S \to G'_{\alpha_2(S)}$ and a unique group homomorphism $\gamma_S: H \cap G_S \to \text{Center}(G'_{\alpha_2(S)})$ such that $\alpha(h) = \beta_S(\varphi_S(h)) \cdot \gamma_S(h)$ for $h \in H \cap G_S$. Our aim then is to paste the data (φ_S, β_S), $S \in S_2(\mathcal{O})$, together to obtain a ring monomorphism $\varphi: \mathcal{O} \to \mathcal{O}'$.

We derive first from α_2 a map $\alpha_4: S_4(\mathcal{O}) \to S_4(\mathcal{O}')$. Then β_S gives us an isomorphism ${}^{(\varphi_S)}G_D \to G_{\alpha_4(D)}$ (if $D \in S_4(\mathcal{O})$, $S \subset D$). This isomorphism can only come from an algebra monomorphism $\varphi_D: D \to \alpha_4(D)$. We show that $\varphi_D \mid S = \varphi_S$ for $S \subset D$, $S \in S_2(\mathcal{O})$. This permits us to normalize the data and then to show that it can be pasted together. Although conceptually simple, this step is quite intricate.

If char $k' = 3$, it is also necessary to handle the case when the Zariski closure of $\alpha(H \cap G_S)$ is not of the form $G'_{S'}$. In this case we consider another k'-group G'' of type G_2 such that there exists a special k-isogeny $\iota: G' \to G''$. Then we apply our previous result to $\iota \circ \alpha$. It will be easy to show that $\iota \circ \alpha$ is "divisible" by Frobenius whence the result follows in full generality. This part is similar to the treatment [12, §] of orlthogonal groups of dimension 5 in characteristic 2.

We conclude with an Appendix in which the pathology of G_2 induced by characteristic 2 is described.

0.5. *Conventions and notation.* We consider only algebras with an identity element 1. The letters $\mathcal{O}$, S, D usually denote an octave algebra, a two-dimensional separable algebra, and a quaternion algebra. The letter N denotes the norm map and Tr the trace map. Thus $N_{S/k}$ and $\mathrm{Tr}_{S/k}$ are the norm and the trace from S to k. For S and D we denote by S^1 and D^1 the group of elements of norm 1. For a subspace $M \subseteq \mathscr{C}$ we denote by $M^\perp$ the orthogonal complement with respect to the canonical scalar product in $\mathscr{C}$. If V is a vector space over k, then $\mathrm{End}_k V$ is the algebra of k-endomorphisms of V and $GL(V) = (\mathrm{End}_k V)^*$. If $H \subseteq GL(V)$, then V^H is the subspace of fixed points of H in V.

By an algebraic group G we understand an affine reduced group scheme over a field. We denote by G° the connected component of G. By $\mathbf{G}_a$, $\mathbf{G}_m$ we denote the additive and the multiplicative group (defined over prime field). By $\mathrm{Lie}\, G$ we denote the Lie algebra of an algebraic group G. If T is a maximal subtorus of an algebraic group G, then $\Sigma = \Sigma(G, T)$ is a root system of G with respect to T. For any subset $\tilde{\Sigma} \subset \Sigma(G, T)$ we denote by $G(\tilde{\Sigma})$ the subgroup of G generated by root subgroups ($\simeq \mathbf{G}_a$) corresponding to roots from $\tilde{\Sigma}$. If X is an algebraic variety defined over k and $\varphi\colon k \to k'$ is a field homomorphism, then ${}^\varphi X$ denotes the algebraic variety over $\varphi(k)$ obtained from X by base change φ and $\varphi^\circ\colon X(k) \to {}^\varphi X(k')$ is the corresponding map of points.

For an algebraic (resp., abstract) group G and an algebraic (resp., abstract) subset M of G we denote by $Z_G(M)$, $N_G(M)$, $\mathscr{D}^i(G)$, and $C(G)$ the algebraic (resp., abstract) subgroup of G which is respectively the centralizer of M in G, the normalizer of M in G, the ith derived group of G, and the center of G. If G is as above and $M_1, M_2, \ldots, M_n$ are algebraic (resp., abstract) subsets of G, then $\langle M_1, \ldots, M_n \rangle$ denotes the algebraic (resp., abstract) group generated by the groups $M_1, \ldots, M_n$.

The prefixed letter P stands for projectivisation. Explicitly, if V is a free module over A, then PV denotes the corresponding projective space (the set of rank 1 direct summands). If $\alpha\colon V \to V'$ is an imbedding of vector spaces, then $P\alpha\colon PV \to PV'$ is the induced mapping of projective spaces. If G is a group, then $PG = G/C(G)$, and if $\alpha\colon G \to G'$ is a group homomorphism such that $\alpha(C(G)) \subseteq C(G')$, then $P\alpha\colon G/C(G) \to G'/C(G')$.

If $\beta\colon X \to Y$ is a map and Z is a subset of X, then $\beta|_Z$ or $\beta \mid Z$ denotes the restriction of β to Z.

Finally, $|X|$ denotes the cardinality of X.

0.6. Warning. There is an ambiguity in our handling of some functors (algebraic groups, algebras, vector spaces). The ambiguity is caused by the fact that classically vector spaces, algebras, etc., are not considered as functors (from fields to algebras, say), whereas algebraic groups are considered as functors. Our approach is: vector spaces, algebras, etc., are functors, but we do not mention this fact and deal with them as if they are points of the corresponding functor over the given field.

1. Cayley Algebras and Their Subalgebras

General references for this section: N. Jacobson [4], T. A. Springer [1, 6].

Let k be an infinite field and let $\mathscr{C}$ be an octave algebra over k. Denote by $S_i(\mathscr{C})$ the set of semi-simple subalgebras of $\mathscr{C}$ of dimension (i). It follows from 1.5 below that $S_i(\mathscr{C})$ is an algebraic variety.

1.1. $\mathscr{C}$ either is split or is a division algebra.

1.2. If $\mathscr{C}$ is split, then $\mathscr{C}$ can be realized as the algebra of vector matrices $\left(\begin{smallmatrix} a & v \\ u & b \end{smallmatrix}\right)$ with $a, b \in k$, $u, v \in k^3$. The product is

$$\begin{pmatrix} a & v \\ u & b \end{pmatrix}\begin{pmatrix} c & x \\ y & d \end{pmatrix} = \begin{pmatrix} ac - v \cdot y & ax + dv + u \wedge y \\ cu + by + v \wedge x & bd - u \cdot x \end{pmatrix},$$

where $v \cdot u = \sum v_i u_i$ if $v = (v_1, v_2, v_3)$, $u = (u_1, u_2, u_3)$, and $u \wedge v$ is determined from the equation $(u \wedge v) \cdot z = \det(u, v, z)$. Our algebra $\mathscr{C}$ has an involution $\left(\begin{smallmatrix} a & v \\ u & b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} b & -v \\ -u & a \end{smallmatrix}\right)$ and the norm $Q\left(\begin{smallmatrix} a & v \\ u & b \end{smallmatrix}\right) = ab + u \cdot v$.

The subalgebra $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), a, b \in k\} \simeq k \oplus k$ is a two-dimensional separable subalgebra. The restriction of Q to S is the norm form of S.

The subalgebra $D = \{\left(\begin{smallmatrix} a & v \\ u & b \end{smallmatrix}\right) \mid a, b \in k, u, v \in ke_1\}$, where $e_1 = (1, 0, 0) \in k^3$, is associative and isomorphic to the algebra of 2×2-matrices; an isomorphism is given by $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right)$. The restriction of Q to D is equivalent to the determinant map of D.

1.3. $\mathscr{C}$ has an involution $x \mapsto \bar{x}$ such that

(i) $\overline{xy} = \bar{y}\bar{x}$.

(ii) $x\bar{x} = \bar{x}x = Q(x) \cdot 1$, where $Q(x)$ is a non-degenerate quadratic form on $\mathscr{C}$ (of defect 0 if char $k = 2$),

(iii) $x(\bar{x}y) = Q(x)y$.

1.4. Let $(x, y) = Q(x + y) - Q(x) - Q(y)$ be the bilinear form associated to Q. Then

(i) $(*, *)$ is non-degenerate,

(ii) $x^2 - (1, x) + Q(x) = 0$ for every $x \in \mathscr{C}$,

(iii) $xy + yx - (1, x)y - (1, y)x + (x, y) = 0$ for all $x, y \in \mathscr{C}$,

(iv) $\bar{x} = (1, x) - x$.

1.5. A subalgebra $M \subseteq \mathscr{C}$ is semi-simple if and only if the restriction of $(*, *)$ to M is non-degenerate.

In this case the set of $d \in M$ such that $Q(d) \neq 0$ is Zariski open in $M^\perp$.

1.6. If $M \subseteq \mathcal{O}$ is a semi-simple subalgebra and $d \in M^{\perp}$ is such that $Q(d) \neq 0$, then

(i) $\dim Md = \dim M$,

(ii) $M \oplus Md$ is a semi-simple subalgebra of $\mathcal{O}$.

1.7. If M is a proper semi-simple subalgebra of $\mathcal{O}$, then

(i) M is associative,

(ii) $\dim M = 1, 2$, or 4,

(iii) if $\dim M = 1$, then $M = k \cdot 1$,

(iv) if $\dim M = 2$, then M is commutative and separable,

(v) if $\dim M = 4$, then M is an algebra of (generalized) quaternions.

1.8. Let $S \in S_2(\mathcal{O})$. For $s_1, s_2 \in S^{\perp}$ set $s_1 s_2 = F_S(s_1, s_2) \oplus m$, where $F_S(s_1, s_2)$ (resp. m) is the orthogonal projection of $s_1 s_2$ onto S (resp. $S^{\perp}$).

PROPOSITION. (i) *$S^{\perp}$ is an S-module.*

(ii) *$F_S \colon S^{\perp} \times S^{\perp} \to S$ is a non-degenerate Hermitian form on $S^{\perp}$.*

(iii) *For $s, t \in S^{\perp}$ we have $F_S(s, \bar{s}) = Q(s)$ and $(s, t) = \mathrm{Tr}_{S/k} F_S(s, \bar{t})$. In particular, orthogonality with respect to F_S implies orthogonality with respect to $(*, *)$.*

Proof. It is sufficient to give the proofs over an extension of k. Then we can assume that $S \simeq k \oplus k$, and, in particular, $\mathcal{O}$ is split. The stabilizer of S in $\mathrm{Aut}_k \mathcal{O}$ contains a split maximal k-torus (cf., 1.2 above), say T, and $S = \mathcal{O}^T$ (the fixed points). Since all split k-tori are conjugate, we can assume that S is the algebra of diagonal matrices in a vector–matrix realization of $\mathcal{O}$ (cf. 1.2). Thus $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), a, b \in k\}$. Then $S^{\perp} = \{\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right), x, y \in k^3\}$. From $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & ax \\ by & 0 \end{smallmatrix}\right)$ we deduce (i) and from $\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & v \\ u & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -x \cdot u & y \wedge u \\ x \wedge v & -y \cdot v \end{smallmatrix}\right)$ we get that $F_S(\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & v \\ u & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} -x \cdot u & 0 \\ 0 & -y \cdot v \end{smallmatrix}\right)$. Thus $F_S(as_1, s_2) = aF_S(s_1, s_2)$ for s_1, $s_2 \in S^{\perp}$, $a \in S$, and $F_S(s_1, s_2) = (F_S(s_2, s_1))^{\sigma}$, where σ is the permutation of diagonal terms, that is, the unique non-trivial k-automorphism of S. This concludes the proof of (ii).

To prove (iii) note that $F_S(s, \bar{s}) = \left(\begin{smallmatrix} x \cdot y & 0 \\ 0 & x \cdot y \end{smallmatrix}\right) = Q(s) \in k$ for $s = \left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)$. The equality $(s, t) = \mathrm{Tr}_{S/k} F_S(s, \bar{t})$ formally follows by definition of $(*, *)$, cf. 1.4.

1.9. LEMMA. *Every element of $\mathcal{O}$ is contained in a quaternion subalgebra of $\mathcal{O}$.*

Proof. Let $y \in \mathcal{O}$. Then $B = k + ky$ is a subalgebra of $\mathcal{O}$. If B is separable, then we take $d \in B^{\perp}$ such that $Q(d) \neq 0$ (cf. 1.5) and set $D = B + Bd$. Then D is quaternion by 1.7 and $D \ni y$ by construction.

Suppose that B is not separable. Then $(*, *) \mid B$ is degenerate and we can assume that $Q(y) = (1, y) = (y, y) = 0$, i.e., $y \in B \cap B^{\perp}$. Since $(*, *)$ is non-degenerate, there exists $x \in \mathcal{C}$ with $(x, y) = 1$. Replacing x by $x - (1, x) \cdot 1 - Q(x) y$ gives $Q(x) = (x, x) = (1, x) = 0$. Consider the subspace $D = k + ky + kx + kxy \subseteq \mathcal{C}$. By 1.4(iii), $yx \in D$. Using (1.4) of [1] and the expression $x = (1, x) - \bar{x}$ we see that $xD \subseteq D$ and similarly $yD \subseteq D$. So D is a subalgebra of $\mathcal{C}$.

Suppose now that char $k \neq 2$. Then $(1, 1) \neq 0$ and it follows from 1.4(ii) that $x^2 = y^2 = 0$. It then follows from (1.1) of [1] that $(xy, yx) + (x^2, y^2) = (x, y)(y, x) = 1$ whence $(xy, yx) = 1$. Since $xy \cdot \bar{y}\bar{x} = 0$ and $yx \cdot \bar{x}\bar{y} = 0$, the plane $kxy + kyx$ is non-degenerate. From $(x, xy) = (1, \bar{x}(xy)) = 0$ and similar equalities we get that the planes $kx + ky$ and $kxy + kyx$ are orthogonal. Therefore $(*, *) \mid D$ is non-degenerate, whence D is quaternion.

If char $k = 2$, then $(1, 1) = 0$ and therefore $B^{\perp} \supset B$ and $(k + kx)^{\perp} \supset k + kx$. We again have that xy is orthogonal to x and y and that $(xy, xy) = 0$. Then $(1, xy) = (\bar{y}, x) = ((1, y) - y, x) = -1$ whence $(*, *) \mid D$ is non-degenerate, i.e., D is quaternion.

1.10. Lemma. *The set of semi-simple subalgebras of dimension i, $i = 1, 2, 4$, is non-empty and Zariski-open in the set of all subalgebras of dimension i.*

Proof. The codition that a subalgebra M is not semi-simple is the condition that $(*, *) \mid M$ is degenerate. It remains to remark that for each $i = 1, 2, 4$ there exists a semi-simple subalgebra of dimension i. It can be constructed using 1.6 repeatedly starting with $M = k$.

1.11. Corollary. (i) *For every $x \in \mathcal{C}$ the set of quaternion subalgebras containing x is non-empty and open in the set of all subalgebras of dimension 4 containing x.*

(ii) *For every $S \in S_2(\mathcal{C})$ the set of $S' \in S_2(\mathcal{C})$ such that S, S' generate a quaternion algebra is non-empty and Zariski-open in the set $S_2(\mathcal{C})$.*

Proof. The first assertion is the combination of Lemmas 1.9 and 1.10. The second also follows from Lemma 1.10 once we have shown that our set is non-empty. Write $S = k + kx$. Find a quaternion algebra $D \ni x$ (by Lemma 1.9) and take a separable subalgebra $S' \subset D$ such that $S' \neq S$. Then S and S' generate D.

1.12. Corollary. *If $S_1, S_2 \in S_2(\mathcal{C})$, then there exists $S_3 \in S_2(\mathcal{C})$ such that S_1, S_3 and S_2, S_3 generate quaternion algebras.*

Proof. Let $R(S)$ be the set of $S' \in S_2(\mathcal{C})$ such that S and S' generate a quaternion algebra. Then $R(S)$ is open and non-empty for any $S \in S_2(\mathcal{O})$. Therefore $R(S_1) \cap R(S_2)$ is open and non-empty. Our assertion follows.

2. Groups of Type G_2 and Their Subgroups

Let k be an infinite field and let G be a group of type G_2, defined over k.

2.1. (Cf., e.g., [1, 4, 10].) There exists an octave algebra $\mathcal{O}$ such that $G = \mathrm{Aut}_k \mathcal{O}$. Conversely, for an octave algebra $\mathcal{O}'$ over k the group $\mathrm{Aut}_k \mathcal{O}'$ is a k-group of type G_2. The group G is split (resp. anisotropic) if $\mathcal{O}$ is split (resp. a division algebra).

2.2. Let T be a maximal subtorus of G. Then

(i) the roots of T in G are $\pm a$, $\pm b$, $\pm(a+b)$, $\pm(2a+b)$, $\pm(3a+b)$, $\pm(3a+2b)$;

(ii) the roots $\pm b$, $\pm(3a+b)$, $\pm(3a+2b)$ are long and the remaining roots are short;

(iii) the weight diagram of the representation of G in $\mathcal{O}/k$ is

$$\begin{aligned} \lambda_1 &= 2a+b \to \lambda_1 - a = a+b \to \lambda - a - b \\ &= a \to \lambda_1 - 2a - b = 0 \to \lambda_1 - 3a - b = -a \to \lambda_1 - 3a - 2b \\ &= -a - b \to \lambda_1 - 4a - 2b = -2a - b. \end{aligned}$$

2.3. Lemma. *Let S be a separable two-dimensional subalgebra of $\mathcal{O}$ and let $\tilde{G}_S$ be the stabilizer of S in G and $G_S = \tilde{G}_S^\circ$. Then*

(i) *G_S is defined over k;*

(ii) *$[\tilde{G}_S : G_S] = 2$, $\tilde{G}_S$ acts on S as* $\mathrm{Aut}_k S$;

(iii) *G_S is a group of type A_2; it acts trivially on S, it acts on $S^\perp$ as* $SU(S^\perp, F_S)$ (cf. 1.8);

(iv) *if T is a maximal torus of G_S, then T is a maximal torus of G and $\Sigma(G_S, T)$ is the subsystem of long roots in $\Sigma(G, T)$.*

Proof. To prove (ii) we use the fact (cf. Corollary 1.11(ii)) that S can be imbedded in a quaternion subalgebra, say D, of $\mathcal{O}$. Then by, Lemma 2.5(i) the group $G^D = \{ g \in \mathrm{Aut}_k \mathcal{O} \mid gD = D \}$ induces on D the full group of automorphisms of D (as an algebraic group). (Since we are giving a survey here which is not self-contained, the forward reference to Lemma 2.5 is lawful.) Therefore there exists an $h \in G^D$ such that $hS = S$ and $h \mid S \neq 1$.

To prove (iii) we can assume (as in 1.8) that $S \simeq k \oplus k$ and that S is the set of diagonal matrices in a vector-matrix realization of $\mathcal{O}$. For $A \in SL(3)$ we define the map $\tilde{A}: \begin{pmatrix} a & x \\ y & b \end{pmatrix} \to \begin{pmatrix} a & Ax \\ {}^tA^{-1}y & b \end{pmatrix}$ of $\mathcal{O}$ into itself; here ${}^tA$ is the transpose of A. We claim that $\tilde{A} \in \mathrm{Aut}_k \mathcal{O}$. First, from the definition of ${}^tA$ we have $(Ax) \cdot y = x \cdot ({}^tAy)$. Now it remains only to check that ${}^tA^{-1}y \wedge {}^tA^{-1}y = A(y \wedge x)$. We have by definition $({}^tA^{-1}y \wedge {}^tA^{-1}x) \cdot z = \det({}^tA^{-1}y, {}^tA^{-1}x, z) = \det({}^tA^{-1}y, {}^tA^{-1}x, {}^tA^{-1}({}^tAz)) = (\det {}^tA^{-1}) \det(y, x, {}^tAz) = (\det {}^tA^{-1})(y \wedge x) \cdot ({}^tAa) = (\det {}^tA^{-1})(A(y \wedge x)) \cdot z$. Since $\det A = 1$, we have that $\tilde{A} \in \mathrm{Aut}_k \mathcal{O}$. So G_S° contains a subgroup of type A_2. Since A_2 is a maximal subgroup of G_2, it follows that G_S is of type A_2. Since G_S acts by automorphisms, it follows that $G_S(S^\perp) \subseteq S^\perp$ and G_S acts by S-endomorphisms on $S^\perp$. Again since $G_S \subseteq \mathrm{Aut}_k \mathcal{O}$, we have $F_S(gs_1, gs_2) = F_S(s_1, s_2)$ for $g \in G_S$. Therefore $\tilde{G}_S \mid S^\perp \subseteq U(S^\perp, F_S)$. Since $\mathrm{rk}\, G_S = 2$, it follows that $\tilde{G}_S \mid S^\perp = SU(S^\perp, F_S)$ as required.

The last assertion (iv) is true since it follows from the weight diagram that only root subgroups corresponding to long roots preserve the fixed space (namely, S) of a maximal torus.

It remains to prove (i). If S is split, this was explicitly shown in the proof of (iii). If S is not split, then we have that G_S is defined over a separable quadratic extension S/k of k and k-closed since $G_S = G \cap \{A \in GL(\mathcal{O}) \mid AS \subseteq S, A \mid S = Id\}$ and both groups in this intersection are defined over k. Thus G_S is defined over k.

2.4. LEMMA. *Let H be a subgroup of type A_2 of G and let T be a maximal torus of H. Then*

(i) *The roots of H with respect to T are either all long or all short. The second case is possible only if* $\mathrm{char}\, k = 3$.

(ii) *If the roots of H are long and H is defined over k, then $H = G_S$ for some $S \in S_2(\mathcal{O})$.*

Proof. The first assertion is known. To prove the second one we take a maximal k-subtorus T of H. Then the subalgebra $S = \mathcal{O}^T$ is defined over k and it follows from the weight diagram that H also acts trivially on S. So it suffices to show that S is separable. But over $\bar{k}$ we have that H is conjugate to a subgroup of the form $G_{\bar{S}}$ for $\bar{S} \in S_2(\mathcal{O} \otimes \bar{k})$ (since they have the same root systems). Thus S is separable.

2.5. LEMMA. *Let D be a quaternion subalgebra of $\mathcal{O}$. Let $G^D = \{g \in \mathrm{Aut}_k \mathcal{O} \mid gD = D\}$ and let $G_D = \{g \in D^D \mid g \mid D = Id\}$.*

(i) *If $x \in D^\perp$ is such that $\mathcal{O} = D \oplus Dx$ (i.e., $Q(x) \neq 0$), then for every $h \in G^D(k)$ there exist $c, q \in D^*$ such that $Q(c) = Q(q)$ and $h(a + bx) = cac^{-1} + (qbc^{-1})x$.*

(ii) *G^D is a connected algebraic k-group of type $A_1 \times A_1$; it is an almost direct (with amalgamated center) product of two k-isomorphic copies of a k-form of $SL(2)$ determined by D (i.e., a k-form H such that $H(k) \simeq \{d \in D \mid Q(d) = 1\}$.*

(iii) *G^D is a maximal algebraic subgroup of G.*

(iv) $G_D(k) \simeq \{d \in D \mid Q(d) = 1\}$.

(v) *The roots of G^D (resp., G_D) with respect to a maximal torus $T \subseteq G^D$ consist of a pair of opposite long roots and a pair of opposite short roots (resp., a pair of opposite long roots).*

Proof. The first assertion follows from Springer [6, pp. 16, 18]. Namely, it is said there that there exist $p, c \in D^*$ with $Q(p) = 1$ such that $h(a + bx) = cac^{-1} + (pcbc^{-1})x$. Setting $q = pc$ we obtain our assertion. Now (ii) follows. Namely, by (i) we have that G^D is defined over k and $G^D(k)$ is the quotient of $S(D^* \times D^*) = \{(x, y) \in D^* \times D^* \mid Q(x) = Q(y)\}$ by the subgroup $\{(x, x) \in k^* \times k^*\}$ of the center. Now (iv) and (v) can be easily verified.

To prove (iii) note that G^D is of maximal rank. Suppose that $H \supset G^D$, $H \neq G$. Then $H^\circ \supset G^D$ and H° must be semi-simple. The only semi-simple subgroups of maximal rank in G are of type A_2 or $A_1 \times A_1$. The case A_2 gives a contradiction between (iv) and Lemma 2.4(i). Thus $H^\circ = G^D$. Therefore $H \subseteq N_G(G^D)$. But then every $h \in H/G^D$, $h \neq 1$, induces a nontrivial automorphism of G^D. By (ii) $\operatorname{Aut} G^D/\operatorname{Ad} G^D \simeq \mathbf{Z}/2$ and a nontrivial automorphism permutes (almost direct) factors of G^D. But h cannot do even that since the lengths of roots of two different factors are different. This concludes the proof of Lemma 2.5.

2.6. LEMMA. *If H is a k-subgroup of type $A_1 \times A_1$ of G, then there exists $\tilde{D} \in S_4(\mathscr{O})$ such that $H = G^{\tilde{D}}$.*

Proof. Since H is of type $A_1 \times A_1$, it contains a maximal k-subtorus T of G. The roots of H must consist (easy analysis of the root system of G) of two orthogonal pairs of opposite roots, a long pair and a short pair. Thus $\tilde{H}$ cannot be almost k-simple. Let $\tilde{H}_1{}'$ be the normal k-subgroup of $\tilde{H}$ corresponding to long roots. Let K be the minimal splitting field of T. Then $\mathscr{O} \otimes K$ is also split and therefore contains a split quaternion algebra D. Then G^D is split over K and therefore K-conjugate to H. Thus the fixed space $\tilde{D}$ of $\tilde{H}_1$ in $\mathscr{O}$ is a quaternion algebra. Let $T_1 = T \cap \tilde{H}_1$. Then $\mathscr{O}^{T_1} = \mathscr{O}^{\tilde{H}_1}$ and since T_1 is defined over k, it follows that $\mathscr{O}^{T_1}$ is defined over k. Thus $\tilde{D} \in S_4(\mathscr{O})$ whence our assertion.

2.7. COROLLARY. *If $S_1, S_2 \in S_2(\mathscr{O})$, then there exists $S_3 \in S_2(\mathscr{O})$ such that $G_{S_1} \cap G_{S_3}$, $G_{S_2} \cap G_{S_3}$, $Z_G(G_{S_1} \cap G_{S_3})$, and $Z_G(G_{S_2} \cap G_{S_3})$ are k-subgroups of type A_1.*

Proof. Given S_1, S_2 find S_3 using Corollary 1.12. Then $G_{S_i} \cap G_{S_3} = G_{D_i}$, $i = 1, 2$, where D_i is the quaternion subalgebra generated by S_i, S_3. Then the rest of the assertion follows from Lemma 2.5(ii).

2.8. PROPOSITION. (i) *If* char $k \neq 3$, *then every k-subgroup $H \subseteq G$ of type A_2 is of the form G_S for some $S \in S_2(\mathscr{C})$.*

(ii) *If* char $k = 3$ *and H, H' are two subgroups of type A_2 such that $H = G_S$, $S \in S_2(\mathscr{C})$ and both $Z_G(H \cap H')$ and $H \cap H'$ are of type A_1, then $H' = G_{S'}$ for $S' \in S_2(\mathscr{C})$.*

Proof. The first assertion follows from Lemma 2.4. Let us prove (ii). Since $\tilde{H} = (H \cap H') \cdot Z_G(H \cap H')$ is of type $A_1 \times A_1$, it follows that $\tilde{H}$ contains a maximal k-torus. Thus $H \cap H'$ is generated by root subgroups and since $H \cap H' \subset G_S$, it follows from 2.2(i) that the corresponding roots are long. Since $H \cap H'$ is contained in H', it follows from Lemma 2.4(i) that the roots of H' are long too. Then by Lemma 2.4(ii) it follows that $H' = G_S$ for $S' \in S_2(\mathscr{C})$, as asserted.

2.9. COROLLARY. *Let H_1, H_2 be two k-subgroups of type A_2 such that $H_1 = G_S$, $S \in S_2(\mathscr{C})$. Then $H_2 = G_{S'}$, $S' \in S_2(\mathscr{C})$, if and only if there exists a k-subgroup H_3 of type A_2 such that $H_1 \cap H_3$, $H_2 \cap H_3$, $Z_G(H_1 \cap H_3)$, and $Z_G(H_2 \cap H_3)$ all are of type A_1.*

Proof. If such an H_3 exists, then by Proposition 2.8(ii) it must be of the form $G_{\bar{S}}$, $\bar{S} \in S_2(\mathscr{C})$. Again, by Proposition 2.8(ii), H_2 is of the form $G_{S'}$, $S' \in S_2(\mathscr{C})$.

The existence of H_3 follows from Corollary 2.7.

2.10. Let Y be the set of singular k-subtori of G.

LEMMA. *Let $T \in Y$. Then $\mathscr{D}Z_G(T)$ is a k-subgroup of type A_1 such that $Z_G(\mathscr{D}Z_G(T))$ is a k-subgroup of type A_1.*

Proof. Since T is singular, it follows that the semi-simple part of $Z_G(T)$ is of type A_1. Therefore $\mathscr{D}Z_G(T)$ is generated by two opposite root subgroups. It then follows from root considerations that $Z_G(\mathscr{D}Z_G(T))$ is also of type A_1.

2.11. COROLLARY. *Any $T \in Y$ is a maximal k-subtorus of the absolutely almost simple k-group $Z_G(\mathscr{D}Z_G(T))$.*

2.12. Call a maximal k-torus of G *admissible* if either it is split over k or it is anisotropic and split over a quadratic extension of k.

LEMMA. (i) *G contains admissible tori,*

(ii) *any k-subgroup of type A_2 contains an admissible torus,*

(iii) *every $T \in Y$ is contained in an admissible torus.*

Proof. Since G always contains subgroups of type A_2 (namely, G_S, $S \in S_2(\mathcal{O})$) we see that (i) follows from (ii). If H is a k-subgroup of type A_2 and $H = G_S$ for some $S \in S_2(\mathcal{O})$, our assertion follows from Lemma 2.3(iii) (cf. [5, 1.3.10]). If H is not of the form G_S, then take a maximal k-torus $T \in H$. Let $\tilde{\Sigma}$ be the subsystem of long roots in $\Sigma(G, T)$. Then $G(\tilde{\Sigma})$ is a k-subgroup of the form G_S, $S \in S_2(\mathcal{O})$. The representation of H on $L = \operatorname{Lie} G/\operatorname{Lie} G(\tilde{\Sigma})$ decomposes over $\bar{k}$ into a direct sum of two three-dimensional representations (as can be seen from the root system of 2.2). Thus H acts on L as a unitary group whence our assertion for H.

Finally, let $T \in Y$. Set $\tilde{G} = \mathscr{D}Z_G(T)$ and $\tilde{G}' = Z_G(\tilde{G})$. Let K be the minimal splitting field of T. Then by, Lemma 2.4(ii), K splits both $\tilde{G}$ and $\tilde{G}'$. Therefore $\tilde{G}$ contains a k-torus T' whose splittig field is K (cf. [11, nos. 1.3.2, 1.2.2]). Then TT' is an admissible torus.

2.13. Let U_r, $r \in \Sigma$, denote a root subgroup of G and let $x_a \colon \mathbf{G}_a \simeq U_r$ be its parametrization. Let us use the notation from 2.2 with T split and defined over k. Then $G(b) \cdot G(2a+b)$ is defined and split over k and is therefore a group G^D for some split quaternion subalgebra $D \in S_4(\mathcal{O})$ (by Lemma 2.6). By Lemma 2.5(i) an element $h_{s,t} = x_b(s)\, x_{2a+b}(t)$ is represented by the automorphism $a + bx \to cac^{-1} + (qbc^{-1})x$ of $\mathcal{O}$ where $c = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in D$ and $q = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \in D$. Writing $a = \left(\begin{smallmatrix} x_2 & x_1 \\ x_4 & x_3 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} y_2 & y_1 \\ y_4 & y_3 \end{smallmatrix}\right)$ we see that $h_{s,t}$ is givcn by the matrix

$$\begin{pmatrix} 1 & -t & t & -t^2 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & -t & s & -st \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

LEMMA. *The elements $h_{s,t}$ and $h_{s',0}$ are not conjugate in G if $t \neq 0$.*

Proof. We have $\dim((h_{s,t} - Id)\mathcal{O}) = 4$ if $t \neq 0$ and $=2$ if $t = 0$, $s \neq 0$. Thus $h_{s,t}$, $t \neq 0$, is not conjugate to $h_{s',0}$ even in $GL(\mathcal{O})$.

3. FULL SUBGROUPS

Let Y be the set of singular k-subtori of G.

3.1. DEFINITION. A subgroup H of $G(k)$ is *full* if $|T \cap H| > 4$ for every $T \in Y$.

Remark. It is quite probable that the condition $|T \cap H| \geqslant 2$ is sufficient for the argument to go through.

3.2. To study full groups we need to record some standard properties of dense subgroups.

LEMMA. *Let G be a connected algebraic k-group and $G_1, G_2, ..., G_m$ closed algebraic k-subgroups of G. Let H be a dense subgroup of G and let H_i be a dense subgroup of G_i.*

(i) *If H_1 is normal in H, then G_1 is normal in G and $H/H \cap G_1$ is dense in G/G_1.*

(ii) *$Z_G(H_1) = Z_G(G_1)$ and $Z_H(H_1) = Z_H(H_1) \cap H$.*

(iii) *$\mathscr{D}^i H$ is dense in $\mathscr{D}^i G$.*

(iv) *If G is connected and reductive, then $\mathscr{D}^i H$ is dense in $\mathscr{D} G$ for all $i \geqslant 1$; in particular, if G is not a torus then $\mathscr{D}^i H \neq 1$ for $i \geqslant 1$.*

(v) *If G_1 is connected and reductive, then $Z_H(\mathscr{D}^i H_1) = Z_H(\mathscr{D} H_1) = H \cap Z_G(\mathscr{D} G_1)$.*

(vi) *$\langle H_1, ..., H_m \rangle$ is dense in $\langle G_1, ..., G_m \rangle$.*

(vii) *If G is not unipotent, H contains a regular element of G of arbitrary large order.*

Proof. In (i) it is clear that G_1 is normal in G. Let $\bar{G}$ be the closure of HG_1/G_1 in G/G_1. Then the preimage of $\bar{G}$ contains H whence $\bar{G} = G/G_1$. Assertion (ii) is clear. Now (iii) follows from (i) if we take G_1 to be the closure of $H_1 = \mathscr{D} H$. Next, (iv) holds because $\mathscr{D} G = \mathscr{D}^i G$ for $i > 1$ and because of (iii). Now (v) is the combination of (ii) and (iv). To prove (vi) note that $H_{i_1} \times H_{i_2} \times \cdots \times H_{i_n}$ is dense in $G_{i_1} \times G_{i_2} \times \cdots \times G_{i_n}$ whence the image of the first product in G is dense in the image of the second. Finally, (vii) holds since the set of regular elements of order $\geqslant n_0$ is open in G and non-empty for any $n_0 \in \mathbf{Z}$.

3.3. DEFINITION. Call a group M *fop-simple* (i.e., simple For Our Purposes) if for every non-central subgroup M_1 of M which is normalized by a non-central normal subgroup of M we have

(i) $\mathscr{D}^m M_1$ is infinite for all $m \geqslant 0$;

(ii) if M_1 contains two commuting normal subgroups, then at least one of them is central.

3.4. LEMMA. *Let R be a connected almost k-simple algebraic k-group and let M be a dense subgroup of $R(k)$. Then*

(i) *the center of M is finite and is contained in the center of R*;

(ii) *M is fop-simple.*

Proof. The first assertion follows from Definition 3.3(i). Let us prove (ii). Let N be a non-central normal subgroup of M. By 3.2(i) its Zariski closure is a non-central normal subgroup of G. It is defined over k since $N \subset G(k)$. Since G is almost k-simple, it follows that N is dense in G. Let M_1 be a non-central subgroup of M normalized by N. Then by 3.2(i), M_1 is dense in G. Now it follows from 3.2(iii) that $\mathscr{D}^m M_1$ is dense in G for all m whence $\mathscr{D}^m M_1 \neq 1$. Suppose that M_1 has two commuting normal subgroups, say N_1 and N_2. Then their Zariski closures are commuting normal k-subgroups of G whence these closures are either central or the whole G. Since both closures cannot be the whole G since they cannot then commute, we have proved our lemma.

3.5. Let us return to our previous assumption that G is of type G_2.

DEFINITION. A subgroup M_1 of a dense group $M \subseteq G(k)$ is called *good* if both M_1 and $Z_M(M_1)$ are fop-simple.

3.6. LEMMA. *Let N be a dense subgroup of G, $N \subseteq G(k)$.*

(i) *Let M be a good subgroup of N and let $\tilde{G}$ be its Zariski-closure. Then $\tilde{G}$ and $Z_G(\tilde{G})$ are connected k-subgroups of type A_1.*

(ii) *If H is a full subgroup of $G(k)$ and $\tilde{G}$ is a connected k-subgroup of type A_1 such that $Z_G(\tilde{G})$ is of type A_1, then $M = \tilde{G} \cap H$ is a good subgroup of H.*

Proof. To prove (i) note first that if M is good, then M is infinite. Then $\tilde{G}^\circ$ is the closure of a normal subgroup of finite index in M. Since M is fop-simple, it follows that $\tilde{G}^\circ$ is not solvable and since $Z_N(M)$ is fop-simple, $Z_G(\tilde{G})$ has the same property. Therefore both $Z_G(\tilde{G})^\circ$ and $\tilde{G}^\circ$ are of type A_1 and by Lemma 2.5(iii) the group $\tilde{G}^\circ \cdot Z_G(\tilde{G})^\circ$ is a maximal subgroup of G. Thus $\tilde{G}$ and $Z_G(\tilde{G})$ are connected. They are defined over k because they are Zariski closures of sets of k-points (M and $Z_H(M)$, respectively).

To prove (ii) note first that $H \cap \tilde{G}$ is dense in $\tilde{G}$ and $H \cap Z_G(\tilde{G})$ is dense in $Z_G(\tilde{G})$ by 2.10. Now our assertion follows from Lemma 3.4.

3.7. LEMMA. *Let H be a full subgroup of $G(k)$ and let G_1 be a connected algebraic k-subgroup of type A_2 in G. Then* $(G_1 \cap H)$/center *is full in* G_1/center (in the sense of [5, §3]).

Proof. Since the center of G_1 does not intersect with any $T \in Y$, $T \subset G_1$, it follows that the image of $T \cap H$ in $(G_1 \cap H)$/center has cardinality >4. If

T_1 is a k-subtorus of G_1 such that $\mathscr{D}Z_{G_1}(T_1)$ is of type A_1, then certainly $T_1 \in Y$. So it remains to show that $|H \cap U| > 1$ for every root k-subgroup U of G_1. Since $Z_{G_1}(U)$ contains a dense set of tori $T' \in Y$, it follows by Lemma 3.2(iii) that $U \supset \mathscr{D}^2(Z_{G_1}(U) \cap H) \neq 1$.

3.8. COROLLARY. *Let H be a full subgroup of $G(k)$ and let G_1 be a connected algebraic k-subgroup of type A_2 in G. Then $G_1 \cap H$ is dense in G_1 and, therefore, $G_1 \cap H$ is fop-simple.*

3.9. LEMMA. *Let H be a full subgroup of $G(k)$. A subgroup M of H is of the form $T \cap H$ for $T \in Y$ if and only if*

(i) *$\mathscr{D}^2 Z_H(M)$ is good,*

(ii) *$M = Z_{\tilde{H}}(M)$ where $\tilde{H} = Z_H(\mathscr{D}^2 Z_H(M))$,*

(iii) *$[N_{\tilde{H}}(M) : M] \leqslant 2$,*

(iv) *M contains an element of order >4.*

Proof. Suppose that $M = T \cap H$, $T \in Y$. Then $|M| > 4$ and since finite subgroups of $T(\bar{k}) = \bar{k}^*$ are all cyclic we have (iv). Denote now $R = Z_G(M)$ and $\tilde{R} = Z_G(\mathscr{D}^2 R)$. Since $M \subset T$, it follows that $R \supseteq Z_G(T)$. So R contains a maximal torus and a subgroup of type A_1 normalized by this torus. It follows that R° is either $Z_G(T)$ or is of type $A_1 \times A_1$ or of type A_2. The centralizer of both of the latter groups does not contain any subgroup of order >3. Thus the only possibility is that $R^\circ = Z_G(T)$. Then it follows from [7, Corollary II.4.4] that $\mathscr{D}^2 Z_G(M) = \mathscr{D} R^\circ$ whence $\mathscr{D}^2 Z_H(M)$ is good by 2.10. Now by Corollary 2.11 we have that $\tilde{R}^\circ = \tilde{R}$ and $\tilde{R}$ is a k-form of $SL(2)$. Therefore $Z_{\tilde{R}}(M) = T$ (by Corollary 2.11 and since the centralizer of any non-central commutative subgroup of a torus in $SL(2)$ is a torus). Since $\tilde{R}$ is a form of $SL(2)$, we have $[N_{\tilde{R}}(T) : T] = 2$ whence (iii).

Now let us prove the converse. Let $\bar{R}$ (resp. $\tilde{R}$) be the Zariski closure of $\mathscr{D}^2 Z_H(M)$ (resp., $Z_H(\mathscr{D}^2 Z_H(M))$). By Lemma 3.6(i) $\bar{R}$ and $\tilde{R}$ are connected commuting k-subgroups of type A_1. Both $\bar{R}$ and $\tilde{R}$ are k-forms of $SL(2)$. Because of (i) and (ii) there are two possibilities: (a) there exists a subtorus T of $\tilde{R}$ such that $T \supset M$ or (b) there exists a unipotent subgroup U of $\tilde{R}$ such that $U \times (\text{center of } \tilde{R}) \supset M$. In the first case $T = Z_{\tilde{R}}(M)$ is defined over k by [2, Proposition 10.3], and therefore $M \subset T$, $T \in Y$. Now consider case (b). Then char $k \neq 2$ by (iv). Therefore U and $N_{\tilde{R}}(U)$ are defined over k (cf. [11, no. 1.4.2]). Thus $N_{\tilde{R}}(U)$ is a Borel k-subgroup of $\tilde{R}$ and therefore $N_{\tilde{R}}(U) \supset \tilde{T}$, $\tilde{T} \in Y$. Thus $[N_H(M) : M] \geqslant |(\tilde{T} \cap H)/(\text{center of } \tilde{R})| > 2$, whence our claim.

4. Initial Study of Monomorphisms

Let k, $\mathscr{O}$, G, etc., be as before. Let k' be another infinite field, let $\mathscr{O}'$ be an octave algebra over k', let $G' = \mathrm{Aut}_{k'} \mathscr{O}'$ be a k'-group of type G_2. Let H be a full subgroup of $G(k)$ and let $\alpha: H \to G'(k')$ be a group momonorphism. For a subset $M \subseteq H$ we denote by $\bar{M}$ the Zariski closure of $\alpha(M)$ in G'.

4.1. Lemma. *If M is a good subgroup of H, then $\bar{M}$ and $Z_{G'}(\bar{M})$ are connected almost simple k-subgroups of G' of type A_1.*

Proof. This follows from Lemma 3.6(i).

4.2. Proposition. *Let T be an admissible subtorus of G. Then $\bar{\alpha}(T) = Z_{G'}(\alpha(T \cap H))$ is a maximal k'-subtorus of G'.*

Proof. Since T is admissible, it contains six distinct singular k-subtori T_i, $i = 1, 2, \ldots, 6$. We choose the numeration in such a way that $[\mathscr{D}Z_G(T_i), \mathscr{D}Z_G(T_{7-i})] = 1$. We have $|T_i \cap T_j| \leqslant 2$ if $i \neq j$. Let $M_i = T_i \cap H$. Then $|M_i| > 4$ (since $T_i \in Y$) and $|M_i \cap M_j| \leqslant 2$ if $i \neq j$.

Since $\mathscr{D}Z_H(M_i)$ is good, it follows that the Zariski closure G_{7-i} of $\alpha(\mathscr{D}Z_H(M_i))$ in G' is a connected k'-group of type A_1 (by Lemma 4.1). We have $G_i = Z_{G'}(G_{7-i})$ (by Lemmas 2.5 and 2.6). In particular, $\alpha(M_i) \subseteq G_i$. Then $\tilde{G}_i = Z_{G_i}(\alpha(M_i))$ is a one-dimensional k-subgroup of G_i (by [11, Sect. 1.5)]. Namely, $\tilde{G}_i$ is either a k-torus or $\tilde{G}_i = U_i \times$ (center of $\tilde{G}_i$) where U_i a unipotent k-subgroup. Set $C_i =$ center of $\tilde{G}_i$. Then $|C_i| \leqslant 2$. Now $G_i \neq G_j$ if $i \neq j$ since otherwise $\alpha(\mathscr{D}Z_H(M_i))$ commutes with $\alpha(\mathscr{D}Z_H(M_{7-j}))$, i.e., $\mathscr{D}_H(M_i)$ commutes with $\mathscr{D}Z_H(M_{7-j})$, i.e., $\mathscr{D}Z_G(T_i)$ commutes with $\mathscr{D}Z_G(T_{7-j})$, which is not the case.

The last remark shows that if $\tilde{G}_i$ is a torus for two values of i, say for s, t, then $T' = \tilde{G}_s \cdot \tilde{G}_t$ is a maximal subtorus of G', $T' \supset \alpha(M)$. Therefore $T' \subseteq Z_{G'}(\alpha(M))$, i.e., $T' = Z_{G'}(\alpha(M))^\circ$. On the other hand, the groups $Z_{G'}(\alpha(M_i))$ are connected (since $Z_{G'}(\alpha(M_i)) = G_{7-i} \cdot Z_{G_i}(\alpha(M_i))$ and by [11, Corollary 1.5.9]). Since $Z_{G'}(\alpha(M)) \subseteq \bigcap_i Z_{G'}(\alpha(M_i)) \supset T'$ we have $T' = Z_{G'}(\alpha(M))$. Thus it only remains to consider the following cases: (a) $\tilde{G}_i$ is a torus for just one value of i and (b) $\tilde{G}_i/C_i$ is unipotent for all i.

In case (a) assume that $\tilde{G}_1$ is a torus. Take then $h \in M_i$, $i \neq 1, 6$, such that $h^2 \notin M_1 \cup M_6$ (which is possible in view of Lemma 3.9(iv) and since $|M_i \cap M_j| \leqslant 2$). Then $\alpha(h) = tu$, $t \in \tilde{G}_1$, $u \in U_6$ (we absorb C_6 into $\tilde{G}_1$). Since $h^2 \notin M_1 \cup M_6$, we have $\alpha(h)^2 \notin \tilde{G}_1 \cup \tilde{G}_6$. Thus $t^2 \neq 1$, $u^2 \neq 1$. Then $Z_{G'}(\alpha(h)) = Z_{Z_{G'}(t)}(u) = \tilde{G}_1 \cdot \tilde{G}_6$ is commutative contradicting the fact that $\mathscr{D}Z_H(M_i)$ is good.

It remains to consider the case when all $\tilde{G}_i/C_i$ are unipotent. The group U_i is a root subgroup corresponding, say, to the root a_i. We can assume that the roots a_1, a_2, a_3 are long and a_4, a_5, a_6 are short. Of course, a_i is orthogonal to a_{7-i}. Thus we can assume that $a_1 = b$ and $a_6 = 2a + b$ (in the notation of 2.2). We parametrize the groups U_b, U_{2a+b} as in 2.13. Now once again take $h \in M_3$, such that $h^2 \notin M_1 \cup M_6$. Then $\alpha(h)^2 = h'_{s,t}$ in the notation of 2.12 (but in G') with $s, t \in k'$, $s \neq 0$, $t \neq 0$. By 2.13, $h'_{s,t}$ cannot be conjugated into U_b. This means that U_{a_3} is not conjugate to U_b, that is, the root a_3 is not long. This is a contradiction.

4.3. THEOREM. *Let $\tilde{G}$ be a connected k-subgroup of type A_2 in G. Then the connected component $\tilde{G}'$ of the Zariski closure of $\alpha(\tilde{G} \cap H)$ is an algebraic k-subgroup of type A_2.*

Proof will be given in two independent steps: the first, an easy one, shows that either $\tilde{G}'$ is of type A_2 or else $\tilde{G}' = G'$. The second step shows that the assumption $\tilde{G}' = G'$ leads to a contradiction.

Note first that $\tilde{G}'$ is the Zariski closure of a subgroup N of finite index in $H \cap \tilde{G}$ (namely, of the subgroup $\alpha^{-1}(\alpha(H \cap \tilde{G}) \cap \tilde{G}'))$. Suppose that $\tilde{G}'$ is neither G' nor of type A_2. Let T be an admissible torus of $\tilde{G}$ (cf. 2.12). Then $G(a)$ is defined over k for every $a \in \Sigma(\tilde{G}, T)$. Take $R = G(a)$ for $a \in \Sigma(\tilde{G}, T)$. Then R and $Q = Z_G(R)$ are k-groups of type A_1. Therefore $R \cap H$ is a good subgroup of H. Thus the Zariski closure R' of $\alpha(R \cap H)$ is a connected k'-subgroup of $\tilde{G}'$ of type A_1 (cf. Lemma 4.1). This shows, in particular, that $\tilde{G}'$ is not solvable. If $\tilde{G}'$ is semi-simple, then it is of type $A_1 \times A_1$ and then $\alpha(R \cap H)$ is a normal subgroup of $\alpha(\tilde{G} \cap H)$, which is not the case. So the only remaining possibility is: $\tilde{G}'$ has a radical P' and $\tilde{G}'/P'$ is of type A_1. Since N is fop-simple, we see that $P' \cap \alpha(N)$ is contained in the center of $\alpha(N)$ and, in particular, $|P' \cap \alpha(N)| \leqslant 3$. On the other hand, $Z_{\tilde{G}'}(R') \subset P'$ and we shall show below that $|Z_{G'}(R') \cap \alpha(N)| > 4$. By Lemma 4.1 the Zariski closure Q' of $Q \cap H$ is a connected k'-subgroup of G'. Therefore $\alpha(Q \cap \tilde{G} \cap H) \subset Q' \cap \tilde{G}'$. But $Q \cap \tilde{G} \in Y$ whence $|Q \cap \tilde{G}| > 4$. Therefore $|Q' \cap \tilde{G}' \cap \alpha(N)| > 4$. In particular $|Z_{G'}(R') \cap \alpha(N)| > 4$ whence $|P' \cap \alpha(N)| > 4$, a contradiction. This concludes the first part of the proof.

Assume now that $\tilde{G}' = G'$. Let V be the set of admissible tori in $\tilde{G}$. By [5, Proposition 1.3.11] every pair $T, \tilde{T} \in V$ can be connected by a sequence of length 4 of associated admissible tori where we may need to insert repetitions. This means that there exist $T = T_0, T_1, T_2, T_3, T_4 = \tilde{T} \in V$ such that $T_i \cap T_{i+1} \in Y$ and (automatically) $T_{i+1} \subset Z_G(T_i \cap T_{i+1})$. Set $T'_i = \tilde{\alpha}(T_i)$ (cf. Proposition 4.2). Then $T'_{i+1} \subset Z_{G'}(T'_i \cap T'_{i+1})$ and $\mathscr{D}Z_{G'}(T'_i \cap T'_{i+1})$ is a k'-subgroup of G' of type A_1 normalized y T'_i. Therefore $\mathscr{D}Z_{G'}(T'_i \cap T'_{i+1}) = G'(b_i)$ for some $b_i \in \Sigma(\tilde{G}, T'_i)$. Therefore there exists $h_i \in G'(b_i)$ such that $T'_{i+1} = h_i T'_i h_i^{-1}$. Then $T'_1 = h_0 T'_0 h_0^{-1}$, $T'_2 = h_1 h_0 T'_0 (h_1 h_0)^{-1}$, $T'_3 = (h_2 h_1 h_0) T'_0 (h_2 h_1 h_0)^{-1}$, and $T'_4 = (h_3 h_2 h_1 h_0) T'_0 (h_3 h_2 h_1 h_0)^{-1}$. Set $\tilde{h}_0 = 1$, $\tilde{h}_{i+1} = h_i \cdots h_0$, $i = 0, 1, 2, 3$. Then $T'_{i+1} = \tilde{h}_{i+1} T_0 \tilde{h}_{i+1}^{-1}$, $i = 0, 1, 2, 3$.

Now take $\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3 \in \Sigma(\tilde{G}', T'_0)$ such that $\tilde{h}_i G'(\tilde{b}_i) \tilde{h}_i^{-1} = G'(b_i)$, $i = 0, 1, 2, 3$ (clearly both groups are normalized by T'_i). There exists $h'_i \in G'(\tilde{b}_i)$ such that $h_i = \tilde{h}_i h'_i h_i^{-1}$. We have $h_0 = h'_0$ and $h_1 = \tilde{h} h' h_1^{-1} = h_0 h'_1 h_0^{-1} = h'_0 h'_1 h'^{-1}_0$, whence $\tilde{h}_2 = h_1 h_0 = h'_0 h'_1$. Continuing in this manner we establish that $\tilde{h}_4 = h'_0 h'_1 h'_2 h'_3$. Let R' denoted the algebraic subset of $\tilde{G}$ which is the union of quadruple products: $R' = \bigcup G'(c_1) G'(c_2) G'(c_3) G'(c_4)$, where $c_1, c_2, c_3, c_4 \in \Sigma(\tilde{G}', T')$. The above argument implies: for every $\tilde{T} \in V$ there exists $r \in R'$ such that $\bar{\alpha}(\tilde{T}) = r\bar{\alpha}(T) r^{-1}$. Thus $\alpha(\tilde{T})$ is contained in $V' = \{r\bar{\alpha}(T) r^{-1}, r \in R'\}$. Let us estimate dim V'. We have $\dim R' \leqslant 10$ (because tori $\bar{\alpha}(T) \cap G'(c_i)$ enter several times). Therefore $\dim V' \leqslant 10$ (again since the tori $\bar{\alpha}(T) \cap G'(c_i)$ do not move $\bar{\alpha}(T)$). Now we have for $n \in N$ that $nTn^{-1} \in V$ and it follows that $\alpha(n) \bar{\alpha}(T)(n)^{-1} \in V'$. Since $\alpha(N)$ is dense in $\tilde{G}'$, it follows that $g\bar{\alpha}(T) g^{-1} \in V'$ for $g \in \tilde{G}'$. But if $\tilde{G}' = G'$, then $\tilde{G}'$ acts transitively on the set of all tori of G' and $\bigcup_{g \in G'} gTg^{-1}$ is an open subset of G' which contradicts $\dim V' \leqslant 10$. This concludes the proof of Theorem 4.3.

4.4. For a k-subgroup $\tilde{G} \subset G$ of type A_2 we denote by $\bar{\alpha}(\tilde{G})$ the connected component of the Zariski closure of $\alpha(H \cap \tilde{G})$ in G'.

COROLLARY. (i) *If* $\operatorname{char} k' \neq 3$, *then there exists a unique map* $\alpha_2 \colon S_2(\mathcal{O}) \to S_2(\mathcal{O}')$ *such that* $\bar{\alpha}(G_S) = G'_{\alpha_2(S)}$

(ii) *Suppose* $\operatorname{char} k' = 3$ *and there exists* $\tilde{S} \in S_2(\mathcal{O})$ *such that* $\bar{\alpha}(G_{\tilde{S}}) \subseteq G'_{S'}$, $S' \in S_2(\mathcal{O}')$, *then there exists a unique map* $\alpha_2 \colon S_2(\mathcal{O}) \to S_2(\mathcal{O}')$ *such that* $\alpha(G_S) \subseteq G'_{\alpha_2(S)}$, $S \in S_2(\mathcal{O})$.

Proof. This is a direct corollary of Theorem 4.3, Corollary 2.9, and Lemma 4.1.

5. Monomorphisms Preserving Lengths of Roots

In addition to the notation and conventions of the beginning of Section 4 we make the following

5.1. *Assumption.* There exists $S \in S_2(\mathcal{O})$ such that $\alpha(G_S \cap H) \subseteq G_{S'}$ for some $S' \in S_2(\mathcal{O})$.

5.2. COROLLARY. (i) *There exists a unique map* $\alpha_2 \colon S_2(\mathcal{O}) \to S_2(\mathcal{O}')$ *such that* $\alpha(G_S \cap H) \subseteq G'_{\alpha_2(S)}$ *for* $S \in S_2(\mathcal{O})$;

(ii) *there exists a unique map* $\alpha_4 \colon S_4(\mathcal{O}) \to S_4(\mathcal{O}')$ *such that* $\alpha(G_D \cap H) \subseteq G'_{\alpha_4(D)}$ *for* $D \in S_4(\mathcal{O})$;

(iii) *the inclusion* $S \subseteq D$ *holds for* $S \in S_2(\mathcal{O})$, $D \in S_4(\mathcal{O})$ *if and only if the inclusion* $\alpha_2(S) \subseteq \alpha_4(D)$ *holds*;

(iv) *for* $h \in H$ *and* $R \in S_i(\mathcal{O})$, $i = 2, 4$, *one has* $\alpha_i(hR) = \alpha(h)\,\alpha_i(R)$.

Proof. The first assertion is contained in 4.4. Let $D \in S_4(\mathcal{O})$ and let $S \in S_2(D)$. Then $\alpha(G_D \cap H) \subseteq G'_{S'}$, where $S' = \alpha_2(S)$. By Lemmas 3.6 and 4.1 the Zariski closure $\tilde{G}'$ of $\alpha(G_D \cap H)$ is of the form $G'_{D'}$ or $Z_{G'}(G'_{D'})$ for a unique $D' \in S_4(\mathcal{O}')$. Since $\tilde{G}' \subseteq G'_{S'}$, the roots of $\tilde{G}'$ are long so that $\tilde{G}' = G'_{D'}$. Setting $\alpha_4(D) = D'$ we obtain (ii) and (iii). The last assertion holds since $G_{hR} = hG_R h^{-1}$ and $\alpha(G_{hR} \cap H) = \alpha(h(G_R \cap H)\,h^{-1}) = \alpha(h)\,\alpha(G_R \cap H)\,\alpha(h)^{-1} \subseteq \alpha(h)\,G'_{\alpha_i(R)}\alpha(h)^{-1} = G'_{\alpha(h)\alpha_i(R)}$, as required.

5.3. *Remark.* Let us call two-dimensional subalgebras of $\mathcal{O}$ "points" and four-dimensional subalgebras which are limits of quaternion subalgebras "lines." Then the above shows that most "lines" are mapped into "lines." This is very similar to the setting of the fundamental theorem of projective geometry. If we would have an analog of this theorem, we would be through. Such an analog is unknown but we actually have more information (cf. Corollary 5.4 below) and we use it to bypass the above question.

5.4. COROLLARY. *There exist a unique monomorphism of rings with involution* $\varphi_S \colon S \to \alpha_2(S) \subseteq \mathcal{O}'$, *a unique (up to multiplication by an element of* $\alpha_2(S)^*$) φ_S*-semilinear monomorphism* $\beta_S \colon S^\perp \to \alpha_2(S)^\perp$ *and a unique group homomorphism* $\gamma \colon G_S \cap H \to S^*$ *such that*

(a) $F_{\alpha_2(S)}(\beta_S(x), \beta_S(y)) = a_S\varphi_S(F_S(x, y))$ *for* $x, y \in S^\perp$ *with* $a_S \in k^*$;

(b) $\beta_S(hx) = \gamma(h^{-1})\,\alpha(h)\,\beta_S(x)$ *for* $x \in S^\perp$, $h \in H \cap G_S$;

(c) $(\gamma(h))^3 = 1$ *for* $h \in G_S \cap H$.

In particular,

(i) *if* $\psi = \varphi_S \mid k$, *then there exists a unique* k'*-isomorphism of* k'*-groups* $\tilde{\beta}_S \colon {}^{\psi}G_S \to G'_{\alpha_2(S)}$ *such that* $\alpha(h) = \gamma(h) \cdot \tilde{\beta}_S(\psi^0(h))$ *for* $h \in G_S \cap H$ *(here* $\gamma(h)$ *is considered as an element of the center of* $G'_{S'}$);

(ii) *if* $D \in S_4(\mathcal{O})$ *and* $S \subset D$, *then* $\alpha_2(S)\,\beta_S(S^\perp \cap D) = \alpha_2(S)^\perp \cap \alpha_4(D)$;

(iii) *for* $h \in H$, *we have* $\varphi_{hS} = \alpha(h) \circ \varphi_S \circ h^{-1}$ *and* $\beta_{hS} = c_{h,S}\,\alpha(h) \circ \beta_S \circ h^{-1}$, *where* $c_{h,S} \in \alpha_2(hS)^*$.

Proof. All assertions except (ii) and (iii) are contained in [5, §4]. Now (iii) follows immediately from the uniqueness properties of β_S and φ_S. So it remains to prove (ii). Set $S' = \alpha_2(S)$ and $D' = \alpha_4(D)$. We know that $S^\perp \cap D$ is the set of fixed points of G_D in $S^\perp$. By (b), $G'_{D'}$ acts on $\beta_S(S^\perp \cap D)$ by multiplication by elements of S'. But $G'_{D'}$ is semi-simple. Therefore it acts trivially on $\beta_S(S^\perp \cap D)$ whence $\beta_S(S^\perp \cap D) \subseteq S'^\perp \cap D'^\perp$. Since $S'\beta_S(S^\perp \cap D)$ is a free S'-submodule of $S'^\perp$, it follows that $\beta_S(S^\perp \cap D) = S'^\perp \cap D'$ as required.

5.5. Let $D \in S_4(\mathcal{O})$ and $D' = \alpha_4(D) \in S_4(\mathcal{O}')$. Let $\delta: G_D(\bar{k}) \to D_k \otimes \bar{k}$ (resp., $\delta': G'_{D'}(\bar{k}') \to D' \otimes_{k'} \bar{k}'$) be a (functorial over extensions) mapping identifying $G_D(\bar{k})$ with $(D \otimes \bar{k})^1$ for $\bar{k} \supseteq k$ (resp., $G'_{D'}(\bar{k}')$ with $(D' \otimes \bar{k}')^1$ for $\bar{k}' \supseteq k'$). Take $S \in S_2(D)$ and set $\psi = \varphi_S \mid k$.

LEMMA. *There exists a unique monomorphism of rings $\varphi(\delta, \delta'): D \to D'$ such that $\delta' \circ \alpha = \varphi(\delta, \delta') \circ \delta$ on $H \cap G_D$. Moreover, $\varphi(\delta, \delta') \mid k = \psi$.*

Proof. Since G_D does not intersect the center of G_S, it follows from Corollary 5.2(ii) and Remark 5.3(b) that $\gamma(G_D \cap H) = \{1\}$. Then $\alpha(h) = \tilde{\beta}_S(\psi^\circ(h))$ for $h \in G_D \cap H$. Replacing D by $\psi^\circ(D)$ we see that α is given by an algebraic isomorphism. Since the set of k'-algebra isomorphisms between $\psi^\circ(D) \otimes_{\psi(k)} k'$ and D' is naturally identified with the set of k'-isomorphisms of algebraic k'-groups ${}^{\psi}G_D$ and $G'_{D'}$, we get our assertion.

5.6. COROLLARY. *For $S, \tilde{S} \in S_2(\mathcal{O})$ we have $\varphi_S \mid k = \varphi_{\tilde{S}} \mid k$.*

Proof. If S, $\tilde{S}$ generate a quaternion algebra D, then our claim follows from the lat statement of 5.5 (applicable to both S and $\tilde{S}$). In the general case we invoke Corollary 1.12.

5.7. *Convention.* We replace k by $\varphi_S(k) \subseteq k'$, $S \in S_2(\mathcal{O})$, and therefore assume henceforth that $\varphi_S \mid k = Id$ for $S \in S_2(\mathcal{O})$.

5.8. For a subset $V \subseteq \mathcal{O}$ denote by $L(V)$ the set of lines $L = kl \subseteq V$, $l \in \mathcal{O}$, such that $Q(L) \neq 0$. A similar notation is used for $\mathcal{O}'$. For $L \in L(V)$ we have by 1.6(ii) that $k + L \in S_2(\mathcal{O})$. The map α_2 induces a map $\alpha_{1,S}$: $L(S^\perp) \to L(\alpha_2(S)^\perp)$ defined by $\alpha_{1,S}(L) = (\alpha_2(k + L) + \alpha_2(S)) \cap \alpha_2(S)^\perp$.

LEMMA. (i) *The map $\alpha_{1,S}(L)$ is well-defined;*

(ii) *$\beta_S(L) = s_L \alpha_{1,S}(L)$ for a unique modulo k'^* element $s_L \in \alpha_2(S)^*$;*

(iii) *$\alpha_{1,S}(sL) = \varphi_S(s)\, \alpha_{1,S}(L)$ for $s \in S$;*

(iv) *$\alpha_{1,hS} = \alpha(h) \circ \alpha_{1,S} \circ h^{-1}$ for $h \in H$; in particular, $\alpha_{1,S}(hL) = \alpha(h)\, \alpha_{1,S}(L)$ for $h \in H \cap G_S$.*

Proof. Set $S' = \alpha_2(S)$ and $L' = (S' + \alpha_2(k + L)) \cap S'^\perp$. By 1.6(ii), $D = S + SL \in S_4(\mathcal{O})$. Set $D' = \alpha_4(D)$. We have S', $\alpha_2(k + L) \subset D'$. Since $\dim_{k'}(\alpha_2(k + L) + S') = 3$, $\dim_{k'}(\alpha_2(k + L) + S' + S'^\perp) = 8$, and $\dim_{k'} S'^\perp = 6$, it follows that $\dim_{k'} L' = 1$. We need to show next that

$Q'(L') \neq 0$. Suppose $Q'(L') = 0$. Then $R' = \alpha_2(k+L) + S'$ is a three-dimensional solvable subalgebra of D'. Set $\tilde{G} = Z_G(G_D)$, $T_1 = \tilde{G} \cap G_S$, and $T_2 = \tilde{G} \cap G_{k+L}$. Then T_1, T_2 are two k-tori of $\tilde{G}$. Since $\dim R' = 3$, it follows that $[\alpha(T_1 \cap H), \alpha(T_2 \cap H)]$ is unipotent and hence commutative. Therefore, $[T_1 \cap H, T_2 \cap H]$ is commutative. This means that $\langle T_1, T_2 \rangle$ is a Borel k-subgroup of $\tilde{G}$. Thus $k + L + S$ is a subalgebra. But then $Q(L) = 0$, a contradiction. This proves (i).

Now (ii) follows since $S'\beta_S(L) = D' \cap S'^{\perp} = S'\alpha_{1,S}(L)$ by Corollary 5.4(iii). Next, (iii) follows from (ii). Indeed, applying (ii) to L and sL and taking into account that $\beta_S(sL) = \varphi_S(s)\,\beta_S(L)$ we get that $s_{sL} \equiv \varphi_S(s)\, s_L \bmod k'^*$, i.e., we can take (by (i)) $s_{sL} = \varphi_S(s)\, s_L$ which is exactly (iii). Finally, (iv) follows from Corollary 5.4(iii).

5.9. PROPOSITION. *Let $D \in S_4(\mathcal{O})$. Then there exists a unique monomorphism of rings $\varphi_D\colon D \to \mathcal{O}'$ such that*

(i) $k'\varphi_D(D) = \alpha_4(D)$,

(ii) $\varphi_D(S) = \alpha_2(S)$ *for* $S \in S_2(D)$,

(iii) $\varphi_D \mid S = \varphi_S$ *for* $S \in S_2(D)$,

(iv) $\varphi_{hD} \circ h = \alpha(h) \circ \varphi_D$.

Proof. Take $L \in L(D^{\perp})$ and $S \in S_2(D)$, and set $L' = \alpha_{1,S}(L)$, $D' = \alpha_4(D)$, and $S' = \alpha_2(S)$. Then $L' \subset S'^{\perp}$ and $F_{S'}(S'^{\perp} \cap D', L') = 0$ since $F(S^{\perp} \cap D, L) = 0$ impies $F_{S'}(\beta_S(S^{\perp} \cap D), \beta_S(L)) = 0$, and because of 5.8(ii), (iii). Thus $L' \subset D'^{\perp}$. Take $0 \neq l \in L$, $0 \neq l' \in L'$. Then for $g \in G_D(k)$, $g' \in G'_{D'}(k')$ there exist (by Lemma 2.5(i)) $d_{g,l} \in D^1$, $d'_{g',l'} \in D'^1$ such that $g(l) = d_{g,l}k$ and $g'(l') = d_{g',l'}l'$. Clearly $d_{g,l}$ depends only on g and L and does not depend on $l \in L$, $l \neq 0$. Thus we have two isomorphisms: $\delta_{D,L}$: $G_D(k) \to D^1$ and $\delta'_{D',L'}\colon G'_{D'}(k') \to D'^1$. By 5.5 there exists a monomorphism $\varphi_{D,L}\colon D \to D'$ such that $\alpha_{1,S}(d_{g,l}L) = \varphi_{D,L}(d_{g,l})\, L'$ for $g \in G_D \cap H$.

Now denote by $G_{D,S}$ (resp., $G'_{D',S'}$) the unique k- (resp., k'-) subtorus of G_D (resp., $G'_{D'}$) such that $\delta_{D,L}(G_{D,S}) \subseteq S^*$ (resp., $\delta'_{D',L'}(G'_{D',S'}) \subseteq S'^*$). We have $|H \cap G_{D,S}| > 4$ whence there exists $h \in H \cap G_D$ such that $hl = sl$, $s^2 \notin S^* - k^*$. Then $\alpha(h)\, L' = \alpha(h)\, \alpha_{1,S}(L) = \alpha_{1,S}(hL) = \alpha_{1,S}(sL) = \varphi_S(s)\, \alpha_{1,S}(L) = \varphi_S(s) \cdot L'$ (we used successively 5.8(iv), 5.8(iii), and the definition of L'). This shows that $\varphi_{D,L}(s) \in \alpha_2(S)$ whence $\varphi_{D,L}(S) \subset \alpha_2(S)$ (since $S'_1L' \cap S'_2L' = L'$ for $S'_1 \neq S'_2 \in S_2(D)$). But there exists at most one monomorphism $D \to D'$ which maps S to $\alpha_2(S)$ for all $S \in S_2(D)$. Thus $\varphi_{D,L}$ does not depend on L. This proves the existence of φ_D together with (i) and (ii). We have shown that $\alpha_{1,S}(hL) = \varphi_S(s)\, L' = \varphi_{D,L}(s)\, L'$ for our choice of h. But then $\varphi_S(s) \in \varphi_{D,L}(s) \cdot k'^*$. Since φ_S is a homomorphism of rings with involution, it follows that $Q'(\varphi_S(s)) = 1$. Of course, we have $Q'(\varphi_{D,L}(s)) = 1$. Thus $\varphi_{D,L}(s) = \pm\varphi_S(s)$. Since both $\varphi_{D,L}$ and φ_S are ring homomorphisms, it follows that $\varphi_{D,L}(s^2) = \varphi_S(s^2)$ and therefore $\varphi_{D,L}(a+bs^2) = \varphi_S(a+bs^2)$ for $a, b \in k$. Since $\varphi_{D,L}$ and φ_S coincide on k (by Corollary 5.6, 5.5) and since $s^2 \notin k$, we get (iii).

Finally, (iv) follows from 5.8(iv) and Corollary 5.4(iii).

5.10. COROLLARY. *Let $S \in S_2(\mathcal{C})$, $L \in L(S^\perp)$.*

(i) *$\alpha_{1,S}(L) = \varphi_D(L)$, where $D = S + SL \in S_4(\mathcal{C})$.*

(ii) *$\alpha_{1,S}(L) = \alpha_2(S)^\perp \cap \alpha_2(k+L)$.*

(iii) *If $\tilde{D} \in S_4(\mathcal{C})$ and $\tilde{D} \supset S, L \in L(\tilde{D}^\perp)$, then $\alpha_{1,S}(L) = \alpha_{1,\tilde{S}}(L)$ for $\tilde{S} \in S_2(\tilde{D})$ and $\alpha_{1,S}(L) \in L(\alpha_4(\tilde{D})^\perp)$.*

Proof. The first assertion is trivial since $\alpha_{1,S}(L)$ is defined in inner terms of D, and φ_D is a homomorphism. Now (ii) is clear since $\alpha_{1,S}(L) = \varphi_D((S^\perp \cap D) \cap (k+L)) = \varphi_D(S^\perp \cap D) \cap \varphi_D(k+L) = \varphi_D(S^\perp \cap D) \cap \alpha_2(k+L) \subseteq \alpha_2(S)^\perp \cap \alpha_2(k+L)$. Let us prove (iii). By definition $\alpha_{1,S}(L) \subseteq \alpha_2(S)^\perp$. Set $M = \tilde{D} \cap S^\perp$. Then $F_S(M, SL) = 0$. Therefore $F'_{\alpha_2(S)}(\beta_S(M), \beta_S(SL)) = 0$. By Corollary 5.2(iii) we have $\beta_S(M) = \alpha_4(\tilde{D}) \cap \alpha_2(S)^\perp$ whence $\beta_S(SL) \subset \alpha_4(\tilde{D})^\perp$. Now 5.8(ii) shows that $\alpha_{1,S}(L) \subset \alpha_4(\tilde{D})^\perp$. Finally, by Corollary 5.10(ii), $\alpha_{1,S}(L) = \alpha_4(\tilde{D})^\perp \cap \alpha_2(k+L) = \alpha_{1,\tilde{S}}(L)$. This concludes the proof of (iii).

5.11. Given a pair $S \subset D$, $S \in S_2(D)$, $D \in S_4(\mathcal{C})$, we set $S' = \alpha_2(S)$ and $D' = \alpha_4(D)$. Then $S'\varphi_D(S^\perp \cap D) = S'\beta_S(S^\perp \cap D)$ (by Proposition 5.9(i) and Corollary 5.4(iii)). Since both φ_D and β_S are φ_S-semilinear on $S^\perp \cap D$, it follows that there exists a unique $s' = s'(S, D) \in S'^*$ such that $\varphi_D \mid S^\perp \cap D = s'\beta_S \mid S \cap D$. Set $\beta_{S,D} = s'\beta_S$, $\psi_{S,D} = \varphi_S \oplus \beta_{S,D}$. Then $\psi_{S,D}\colon \mathcal{C} \to \mathcal{C}'$ is a φ_S-semilinear map.

THEOREM. (i) *$\psi_{S,D}$ is a homomorphism of algebras.*

(ii) *$\psi_{S,D}$ does not depend on $D \in S_4(\mathcal{C})$, $S \in S_2(D)$.*

Proof will be given in several steps. First we show (in Lemmas 5.11.1, 5.11.2, and 5.11.3) that $\beta_{S,D}$ agrees with $\alpha_{1,S}$. The difficulty here is that $\alpha_{1,S}$ does not have to be "algebraic" and therefore the action of the group H (which gives dense sets) does not help much. Once we have shown that $\beta_{S,D}(L) = \alpha_{1,S}(L)$ for $L = L(S^\perp)$ it is comparatively easy to show that $\beta_{S,D}$ fit together for different S and D (which is done in Lemma 5.11.4 and 5.11.5).

5.11.1. LEMMA. *There exists $s_{S,D} \in S'^*$ such that $\beta_{S,D}(L) = s_{S,D}\alpha_{1,s}(L)$ for all $L \in L(D^\perp)$; $s_{S,D}$ is unique modulo k'^*.*

Proof. Let $\tilde{S} \in S_2(D)$. We have $k'\beta_{S,D}(D^\perp) = D'^\perp = k'\beta_{\tilde{S},D}(D^\perp)$. Since both $\beta_{S,D}$ and $\beta_{\tilde{S},D}$ are monomorphisms, this implies that there exists

$A_{\tilde{S},S} \in (\mathrm{End}_{k'} D'^{\perp})^*$ such that $\beta_{\tilde{S},D} = A_{\tilde{S},S}\beta_{S,D}$ on $D^{\perp}$. Write $\beta_{S,D}(L) = s_L \alpha_{1,S}(L)$ and $\beta_{\tilde{S},D}(L) = \tilde{s}_L \alpha_{1,\tilde{S}}(L)$ for $L \in D^{\perp}$ (in the notation of 5.8(ii)). Now by Corollary 5.10(iii) we have $\alpha_{1,S}(L) = \alpha_{1,\tilde{S}}(L)$. Thus $A_{\tilde{S},S}(\alpha_{1,S}(L)) = \tilde{s}_L s_L^{-1} \alpha_{1,S}(L)$. Take $h \in G_D \cap H$. Then $s_{hL}\alpha_{1,S}(hL) = \beta_{S,D}(hL) = \alpha(h)\beta_{S,D}(L) = s_L \alpha(h)\alpha_{1,S}(L) = s_L \alpha_{1,S}(hL)$ whence $s_{hL} \in k'^* s_L$ for $h \in G_G \cap H$. Similarly $\tilde{s}_{hL} \in k'^* s_L$ for $h \in G_D \cap H$. This implies that $A_{\tilde{S},S}(\alpha(h)\alpha_{1,S}(L)) = \tilde{s}_L s_L^{-1}\alpha(h)\alpha_{1,S}(L)$. Since $A_{\tilde{S},S}$ is k'-linear, it is determined up to a constant factor by its action on a Zariski-dense set of lines of $D'^{\perp}$. Since $\alpha(G_D \cap H)$ is dense in $G'_{D'}$, it follows that the lines $\alpha(h)\alpha_{1,S}(L)$, $h \in G_D \cap H$, form a dense set. Thus $A_{\tilde{S},S}(\alpha_{1,S}(\tilde{L})) = \tilde{s}_L s_L^{-1}\alpha_{1,S}(\tilde{L})$ for any $\tilde{L} \in L(D^{\perp})$. Thus $\tilde{s}_L^{-1}\beta_{\tilde{S},D}(\tilde{L}) = s_L^{-1}\beta_{S,D}(\tilde{L})$ for all $\tilde{L} \in L(D^{\perp})$. Since $\beta_{S,D}(\tilde{L}) \subset S'^* \cdot \alpha_{1,S}(\tilde{L})$ and $\beta_{\tilde{S},D}(\tilde{L}) \subset \tilde{S}'^*\alpha_{1,\tilde{S}}(\tilde{L})$, it follows from $S'^*\tilde{L}' \cap \tilde{S}'^*\tilde{L}' = \tilde{L}'$ (if $\tilde{S}' \neq S'$) and from Corollary 5.10(iii) that $\beta_{S,D}(\tilde{L}) = s_L \alpha_{1,S}(\tilde{L})$. This proves the existence of $s_{S,D}$. The uniqueness is evident.

5.11.2. LEMMA. *$s_{S,D} \in k'^*$ (i.e., $\beta_{S,D}(L) = \alpha_{1,S}(L)$ for $L \in L(D^{\perp})$).*

Proof. Take $\tilde{L} \in L(D^{\perp})$. Then $\tilde{D} = S + S\tilde{L} \in S_4(\mathscr{C})$. Then $\beta_{S,\tilde{D}} = a s_{S,D}^{-1}\beta_{S,D}$, $a \in k'^*$, since we must have $\beta_{S,\tilde{D}}(\tilde{L}) = \alpha_{1,S}(\tilde{L})$ by the definition of $\beta_{S,\tilde{D}}$ and since both $\beta_{S,D}$ and $\beta_{S,\tilde{D}}$ are linear. Now $\tilde{D}^{\perp} = (D^{\perp} \cap \tilde{D}^{\perp}) \oplus (S^{\perp} \cap D)$. Take $L \in L(D^{\perp} \cap \tilde{D}^{\perp})$. Then $\beta_{S,\tilde{D}}(L) = s_{S,D}^{-1}\beta_{S,D}(L) = \alpha_{1,S}(L)$ (since $D^{\perp} \cap \tilde{D}^{\perp} \subset \tilde{D}^{\perp}$). On the other hand, since $S^{\perp} \cap D \subset \tilde{D}^{\perp}$ for $L \in L(S^{\perp} \cap D)$, we have $\beta_{S,\tilde{D}}(L) = s_{S,D}^{-1}\beta_{S,D}(L) = s_{S,D}^{-1}\alpha_{1,S}(L)$. By Lemma 5.11.1 there exists $s_{S,\tilde{D}} \in S'^*$ such that $s_{S,\tilde{D}} \cdot 1 = s_{S,\tilde{D}} \cdot s_{S,D}^{-1} \bmod k'^*$. Thus $s_{S,D} \in k'^*$ as required.

5.11.3 LEMMA. *$\beta_{S,D}(L) = \alpha_{1,S}(L)$ for $L \in L(S^{\perp})$.*

Proof. By Lemma 5.11.2 we know that $\beta_{S,D}(L) = \alpha_{1,S}(L)$ for $L \in L(D^{\perp})$. Let $L \in L(S^{\perp})$. Let e_1, e_2, e_3 be an orthogonal with respect to F_S basis (over S) of $S^{\perp}$ such that $Se_1 = D \cap S^{\perp}$ and let $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ be an orthogonal basis of $S^{\perp}$ such that $k\tilde{e}_3 = \tilde{L}$ (such bases exist by [5, proof of 1.1.3]). Let e_{1i}, e_{2i}, e_{3i} $(i = 0, 1, 2, 3, 4)$ be a sequence of associated (cf., [5, no. 1.1.4]) bases connecting the basis $\{e_j\}$ to the basis $\{\tilde{e}_j\}$. (We recall that two orthogonal bases in $S^{\perp}$ are associated if they have one element in common.) We have $e_{j0} = e_j$, $e_{j4} = \tilde{e}_j$, $j = 1, 2, 3$. Let $D_i = S + Se_{1i}$. Then $D_i^{\perp} \cap D_{i+1}^{\perp} = D_i^{\perp}$ if $e_{1i} = e_{1i+1}$, and $D_i^{\perp} \cap D_{i+1}^{\perp} = Se_{ji}$ if $e_{ji} = e_{ji+1}$, $j \neq 1$. Thus $L(D_i^{\perp} \cap D_{i+1}^{\perp}) \neq \varnothing$ for all $i = 0, 1, 2, 3$. Let $L_i \in L(D_i^{\perp} \cap D_{i+1}^{\perp})$. Then $\beta_{S,D_i}(L_i) = \beta_{S,D_{i+1}}(L_i) = \alpha_{1,S}(L_i)$ (by Lemma 5.11.2). Consider $\bar{D}_i = S + SL_i$. By Corollary 5.10(i) and by the definition of $\beta_{S,\bar{D}_i}$ we have $\beta_{S,\bar{D}_i}(L_i) \doteq \alpha_{1,S}(L_i)$. Since $\beta_{S,D_i}(L_i) = \beta_{S,D_{i+1}}(L_i) = \alpha_{1,S}(L_i)$, it follows from the definition of β_{S,D_i} that $\beta_{S,\bar{D}_i} = a_i\beta_{S,D_i} = b_i\beta_{S,D_{i+1}}$ with $a_i, b_i \in k'^*$. Thus $\beta_{S,D_{i+1}} = c_i\beta_{S,D_i}$ with $c_i \in k'^*$, $i = 0, 1, 2, 3$. Therefore

$\beta_{S,D}=c\beta_{S,D_4}$. Since $\tilde{L}\in L(D_4^{\perp})$ by construction, it follows from Lemma 5.11.2 that $\beta_{S,D}(\tilde{L})=c\beta_{S,D_4}(\tilde{L})=c\alpha_{1,S}(\tilde{L})=\alpha_{1,S}(\tilde{L})$ as asserted.

5.11.4. LEMMA. (i) *$\psi_{S,D}=\psi_{\tilde{S},D}$ for any $\tilde{S}\in S_2(D)$.*

(ii) *$\psi_{S,D}\colon \mathcal{O}\to\mathcal{O}'$ is a monomorphism of algebras.*

Proof. Clearly, $\psi_{S,D}\mid D=\psi_{\tilde{S},D}\mid D=\varphi_D$ (by construction). On the other hand, $\psi_{S,D}(L)=\alpha_{1,S}(L)=\alpha_{1,\tilde{S}}(L)=\psi_{\tilde{S},D}(L)$ for $L\in D^{\perp}$, $\tilde{S}\in S_2(D)$ (by Corollary 5.10(iii) and Lemma 5.11.2). Thus $\psi_{\tilde{S},D}=\varphi_D\oplus a(\tilde{S},S)(\psi_{S,D}\mid D^{\perp})$ with $a(\tilde{S},S)\in k'^*$. Since $F_{S'}(\psi_{S,D}(l),\overline{\psi_{S,D}(\bar{l})})=\varphi_D(Q(l))=\varphi_S(\overline{F_S(l,\bar{l})})$ for $l\in D\cap S^{\perp}$, it follows from Corollary 5.4(a) that $F_{S'}(\psi_{S,D}(l),\overline{\psi_{S,D}(\bar{l})})=\varphi_S(F_S(l,\bar{l}))$ for all $l\in S^{\perp}$. Since the same holds for $\tilde{S}$ and since $F_S(l,\bar{l})=Q(l)=F_{\tilde{S}}(l,\bar{l})$ for $l\in D^{\perp}$, it follows that $a(\tilde{S},S)$ has norm 1, i.e., $a(\tilde{S},S)=\pm 1$. This proves the first claim if char $k'=2$. Write $S_2(D)=A^+(\bar{S})\cup A^-(\bar{S})$ with $A^{\pm}(\bar{S})=\{\tilde{S}\in S_2(D)\mid a(\tilde{S},\bar{S})=\pm 1\}$. Since we evidently have $a(S_1,S_2)=a(S_1,S_3)\,a(S_3,S_2)$ for $S_1,S_2,S_3\in S_2(D)$, it follows that the function $a(S_1,S_2)$ has value 1, -1, 1 if $(S_1,S_2)\in A^+(\bar{S})\times A^+(\bar{S})$, $A^+(\bar{S})\times A^-(\bar{S})$, $A^-(\bar{S})\times A^-(\bar{S})$, respectively. Thus the partition $S_2(D)=A^+(\bar{S})\cup A^-(\bar{S})$ has a meaning independent of $\bar{S}$ (although signs do depend on $\bar{S}$). We write $S_2(D)=R_1\cup R_2$, $R_1\cap R_2=\varnothing$, $R_1=A^+(\bar{S})$, and $R_2=A^-(\bar{S})$. We have $\psi_{\tilde{S},D}=\psi_{\bar{S},D}$ if both $\bar{S}$ and $\tilde{S}$ are in R_1 (resp., in R_2).

Consider the group $N=Z_G(G_D)$. Any torus of N has the form $T_{\bar{S}}=N\cap G_{\bar{S}}$, $\bar{S}\in S_2(D)$. Since the group $H\cap N$ is dense in N (by Lemma 3.6(ii)) there exists a semi-simple $h\in H\cap N$ such that $h^6\neq 1$ (by Lemma 3.2(vii)). Set $m=h^3$. Suppose $h\in T_{\bar{S}}$, $\bar{S}\in S_2(D)$. Then by Corollary 5.4(b), $\beta_{\bar{S},D}=\gamma(h^{-1})\,\alpha(h)\,\beta_{\bar{S},D}h^{-1}$ whence $\beta_{\bar{S},D}=\alpha(h^3)\,\beta_{\bar{S},D}h^{-3}$ by Corollary 5.4(c). Therefore $\psi_{\bar{S},D}=\alpha(m)\,\psi_{\bar{S},D}m^{-1}$. Since $\psi_{\bar{S},D}=\psi_{\tilde{S},D}$ for $\tilde{S}\in A^+(\bar{S})$, we have $\psi_{\tilde{S},D}=\alpha(m)\,\psi_{\tilde{S},D}m^{-1}$. On the other hand, $\beta_{m\tilde{S},D}=c_{m,\tilde{S}}\alpha(m)\,\beta_{\tilde{S},D}m^{-1}$ with $c_{m,\tilde{S}}\in\alpha_2(m\tilde{S})^*$. Thus $\psi_{m\tilde{S},D}=\alpha(m)[\varphi_{\tilde{S}}\oplus c_{m,\tilde{S}}\beta_{\tilde{S},D}]\,m^{-1}$. Since $\psi_{m\tilde{S},D}\mid D=\psi_{\tilde{S},D}\mid D=\varphi_D$, we have $c_{m,\tilde{S}}=1$ whence $\psi_{m\tilde{S},D}=\alpha(m)\,\psi_{\tilde{S},D}m^{-1}$ and by the above $\psi_{m\tilde{S},D}=\psi_{\tilde{S},D}$. Thus $m(A^+(\bar{S}))=A^+(\bar{S})$ whence $m(A^-(\bar{S}))=A^-(\bar{S})$. Therefore $mR_i=R_i$, $i=1,2$. Let M be the subgroup of N generated by h^3, $h\in H\cap N$, $h^6\neq 1$. By Lemma 3.2(vii), M is dense in N and by the above the partition $S_2(D)=R_1\cup R_2$ is invariant under M.

Set $\bar{M}_i=M\cap\bigcup_{\tilde{S}\in R_i}T_{\tilde{S}}$ and let M_i be the subgroup of M generated by $\bar{M}_i$. We claim that $M_1\cap M_2$ is non-central in M. Since both sets of tori $\bigcup_{\tilde{S}\in R_i}T_{\tilde{S}}$ are dense in the set of tori of N (since both are invariant under conjugation by M), it follows that there exist semi-simple $h_1\in M_1$, $h_2\in M_2$ such that h_1h_2 is semi-simple and $h_1^2\neq 1$, $h_2^2\neq 1$. Then h_1h_2 belongs to one of the $\bar{M}_i$, say to M_1. Thus $h_1h_2\in M_1$. Since $h_1\in M_1$, it follows that

$h_2 = h_1^{-1} \cdot h_1 h_2 \in M_1$. Since $h_2^2 \neq 1$, it follows that h_2 is not central. Thus the group $M_1 \cap M_2$ is not central.

It is clearly normalized by M (since each M_i is) and therefore it is dense in N.

Consider now the groups $P_i = \langle h^3, h \in H \cap G_{\bar{S}}, \bar{S} \in R_i \rangle$. As before we have $\psi_{\bar{S},D} = \alpha(m)\, \psi_{\bar{S},D} m^{-1}$ for $m \in P_i$, $\bar{S} \in R_i$ (by Corollary 5.4(b)). The groups P_i are dense in G and the groups $\alpha(P_i)$ are dense in G.

Let us show that $\psi_{m\bar{S},mD} = \psi_{\bar{S},D}$ for $m \in P_i$, $\bar{S} \in R_i$. Write $D \cap \bar{S}^{\perp} = \bar{S}L$ with $L = kl$, $Q(l) \neq 0$. Then $hD = \bar{S} + \bar{S}(hl)$ for $h \in H \cap G_{\bar{S}}$. By the definition of $\beta_{\bar{S},hD}$ we have $\beta_{\bar{S},hD}(hl) = \varphi_{hD}(hl)$. By Proposition 5.9(iv) we have $\varphi_{hD} = \alpha(h) \circ \varphi_D \circ h^{-1}$. Thus $\beta_{\bar{S},hD}(hl) = \alpha(h)\, \varphi_D(l) = \alpha(h)\, \beta_{\bar{S},D}(l)$. Thus (by the unicity of $\beta_{\bar{S},D}$), $\beta_{\bar{S},hD} = \alpha(h) \circ \beta_{\bar{S},D} \circ h$. By Corollary 5.4(b), $\alpha(h) \circ \beta_{\bar{S},D} \circ h^{-1} = \gamma(h)\, \beta_{\bar{S},D}$, $(\gamma(h))^3 = 1$. Thus for $m = h^3$ we have $\alpha(m) \circ \beta_{\bar{S},D} \circ m^{-1} = \beta_{\bar{S},D}$, $\beta_{\bar{S},mD} = \beta_{\bar{S},D}$. This proves our claim.

Now consider $\tilde{P}_i = \{h \in H \mid \psi_{h\bar{S},hD} = \psi_{\bar{S},D}\}$ for $S \in R_i$. By the above, $\tilde{P}_i \supset P_i$ and $\tilde{P}_1 \cap \tilde{P}_2$ is normalized by $\tilde{P}_1$ and $\tilde{P}_2$. Thus $\tilde{P}_1 \cap \tilde{P}_2$ is either dense or trivial. By the construction of M_i we have $M_1 \cap M_2 \subseteq \tilde{P}_1 \cap \tilde{P}_2$. Thus $\tilde{P} = \tilde{P}_1 \cap \tilde{P}_2$ is dense in G and $\alpha(\tilde{P})$ is dense in G'.

Define $g \in \mathrm{Aut}_{k'} \mathscr{O}'$ by $g(a + bx) = a - bx$ for $a, b \in D'$, $x \in D'^{\perp}$, $Q'(x) \neq 0$ (it is the automorphism of Lemma 2.5(i) with $c = -1 \in D'$, $q = 1 \in D'$). Set $\psi_{\bar{S},D} = \psi_i$ for $\bar{S} \in R_i$. Then $\psi_1 = g \circ \psi_2$.

We assume that $g \neq 1$ (i.e., that char $k \neq 2$) and will arrive at a contradiction. We have $\alpha(m) \circ \psi_i \circ m^{-1} = \psi_i$ for $m \in \tilde{P}$. Since $\alpha(\tilde{P})$ is dense in G', it follows that $g \in Z_{GL(\mathscr{O}')}(G')$. Since $g \in G'$, it means that $g \in C(G')$, i.e., $g = 1$, a contradiction. This concludes the proof of (i).

Let us prove (ii). We know that $\psi_{mS,mD} = \psi_{S,D}$ for $m \in P = \langle h^3 \mid h \in H \cap G_{\tilde{S}}, \tilde{S} \in S_2(D) \rangle$. Since P is dense in G, there are $m_1, m_2, \ldots, m_8 \in P$ such that $\sum_{1 \leqslant i \leqslant 8} m_i S = \mathscr{O}$. Set $S_i = m_i S$, $D_i = m_i D$. Take $y, x \in \mathscr{O}$ and write $x = \sum_{1 \leqslant i \leqslant 8} s_i$, $s_i \in S_i$. Then $\psi_{S,D}(xy) = \sum \psi_{S,D}(s_i y)$ $= \sum \psi_{S_i,D_i}(s_i y) = \sum \varphi_{S_i}(s_i)\, \psi_{S_i,D_i}(y) = [\sum \varphi_{S_i}(s_i)]\, \psi_{S,D}(y) = [\sum \psi_{S_i,D_i}(s_i)]\, \psi_{S,D}(y) = [\sum \psi_{S,D}(s_i)]\, \psi_{S,D}(y) = \psi_{S,D}(\sum s_i)\, \psi_{S,D}(y) = \psi_{S,D}(x)\, \psi_{S,D}(y)$. We used successively: linearity of $\psi_{S,D}$; equality $\psi_{S_i,D_i} = \psi_{S,D}$, $i = 1, \ldots, 8$; φ_{S_i}-semi-linearity of ψ_{S_i,D_i}; equality $\psi_{S,D} = \psi_{S_i,D_i}$, $i = 1, \ldots, 8$; linearity of $\psi_{S,D}$. This concludes the proof of (ii).

5.11.5. *Conclusion of the proof of the Theorem* 5.11. It remains to show that $\psi_{S,D}$ does not depend on D. Suppose $D, \tilde{D} \supset S$, $D, \tilde{D} \in S_4(\mathscr{O})$. Then $\beta_{S,\tilde{D}} = c_{\tilde{D},D} \beta_{S,D}$, where $c_{\tilde{D},D} \in S'^*$ (by Corollary 5.4). Since $\beta_{S,D}(L) = \alpha_{1,S}(L) = \beta_{S,\tilde{D}}(L)$ for $L \in L(S^{\perp})$ (by Lemma 5.11.3) it follows that $c_{\tilde{D},D} \in k'^*$. Since $\psi_{S,D}$ and $\psi_{S,\tilde{D}}$ are both monomorphisms of algebras, it follows that $A_{\tilde{D},D} = Id_{S'} \oplus c_{\tilde{D},D} \cdot Id_{S'^{\perp}} \in \mathrm{Aut}_{k'} \mathscr{O}$. But then $A_{\tilde{D},D} \in G'_{S'}$ and $A_{\tilde{D},D} \in c(G'_{S'})$ whence $c^3_{\tilde{D},D} = 1$.

Assume now that $D^{\perp} \cap \tilde{D}^{\perp}$ is a free S-module, i.e., there exists

$l \in D^{\perp} \cap \tilde{D}^{\perp}$ such that $Q(l) \neq 0$. Since $\psi_{S,D}$ and $\psi_{S,\tilde{D}}$ are algebra monomorphisms, we have $Q'(\psi_{S,D}(l)) = \psi_{S,D}(Q(l))$. But $\psi_{S,D}(Q(l)) = \psi_{S,\tilde{D}}(Q(l))$ since $k \cdot 1 \subset S$. Thus it follows that $Q'(\psi_{S,D}(l)) = Q'(\psi_{S,\tilde{D}}(l))$ which implies that $c^2_{\tilde{D},D} = 1$. Therefore $c_{\tilde{D},D} = 1$ in the case under consideration ($D^{\perp} \cap \tilde{D}^{\perp}$ is a free S-module).

The general case is reduced to the one above in the same manner as in the proof of Lemma 5.11.3 (i.e., using association of orthogonal bases).

5.12. THEOREM. *In the notation and assumptions of this section there exists a unique monomorphism of rings* $\varphi: \mathcal{O} \to \mathcal{O}'$ *such that* $\varphi(hx) = \alpha(h)\,\varphi(x)$ *for* $x \in \mathcal{O}$, $h \in H$. *In particular, there exists a field monomorphism* $\varphi \mid k: k \to k'$ *and a* k'*-isomorphism of algebraic groups* $\beta: {}^{\varphi|k}G \to G'$ *such that* $\alpha = \beta \circ (\varphi \mid k)^{\circ}$.

Proof. Set $\varphi = \psi_{S,D}$, $D \in S_4(\mathcal{O})$, $S \in S_2(D)$. By Theorem 5.11, φ is an algebra monomorphism and does not depend on the choice of S and D. Set $M = \langle \mathscr{Q}(H \cap G_S), S \in S_2(\mathcal{O}) \rangle$. By 5.11.5 we have $\varphi \circ h = \alpha(h) \circ \varphi$ for $h \in M$. Since $M \lhd H$, it now follows from [3, Lemma 5.8] and from the fact that the center of G' is trivial that $\varphi \circ h = \alpha(h) \circ \varphi$ for all $h \in H$. This concludes the proof.

6. MONOMORPHISMS INVERTING LENGTHS OF ROOTS

Now we are going to complete the proof of our main theorem by treating the case when $\alpha(G_S \cap H)$, $S \in S_2(\mathcal{O})$, is not of the form G_S with $S' \in S_2(\mathcal{O}')$. To do this we do not need anymore to represent our group as the group Aut $\mathcal{O}$, $\mathcal{O}$ octaves. Rather we use structure results of algebraic group theory.

6.1. *Assumption.* If T is an admissible subtorus of G, then $\alpha(G(\Sigma_l) \cap H) \subseteq G'(\Sigma'_s)$ where Σ_l is the subset of long roots in $\Sigma = \Sigma(G, T)$ and Σ'_s is the subset of short roots in $\Sigma' = \Sigma(G', \bar{\alpha}(T))$, $\bar{\alpha}(T)$ from Peoposition 4.2.

6.2. *Remark.* By Corollary 4.4 this can happen only if char $k' = 3$. By Lemma 2.4 and 4.4 if the assumption holds for one admissible T, then it holds for all of them.

6.3. Let T be an admissible torus of G, let $T' = \bar{\alpha}(T)$ be the Zariski closure in G' of $\alpha(T \cap H)$ (cf. Proposition 4.2). We apply to the pair (G', T') the construction in [3, no. 3.8] of a non-trivial special k'-isogeny from the k'-group G' to another k'-group G'' of type G_2. The resulting isogeny we denote by $\iota: G' \to G''$ (denoted β_0 in [3, no. 3.8]). Set $T'' = \iota(T')$. Since ι inverts lengths of roots, it follows that $\alpha_1 = \iota \circ \alpha: H \to G''(k')$ preserves the lengths of roots so that α_1 satisfies Assumption 5.1. Thus Theorem 5.12 is applicable to α_1 and there exist a unique field monomorphism $\varphi_1: k \to k'$ and a unique k'-isomorphism $\beta_1: {}^{\varphi_1}G \to G''$ such that $\alpha_1(h) = \beta_1(\varphi_1^\circ(h))$ for $h \in H$.

6.4. At this point let us extract "abstract essence" (R. Steinberg's term). Let k and k' be fields.

Let G be an algebraic absolutely simple k-group, let H be a subgroup of $G(k)$, let G' and G'' be absolutely simple algebraic k'-groups, and let $\iota: G' \to G''$ be a non-central special k-isogeny. Let $\alpha: H \to G'(k')$ be a group homomorphism and suppose that $\alpha_1 = \iota \circ \alpha: H \to G''(k')$ can be expressed in the form $\alpha_1(h) = \beta_1(\varphi_1^\circ(h))$, $h \in H$, with a *unique* field homomorphism $\varphi_1: k \to k'$ and a *unique* k'-isomorphism of algebraic k'-groups $\beta_1: {}^{\varphi_1}G \to G''$. Suppose, in addition, that there is a semi-simple k-subgroup $\tilde{G}$ of G such that

(a) $\tilde{G} \subset G(\Sigma_l)$, where $G(\Sigma_l)$ is the subgroup generated by long root subgroups with respect to some maximal k-torus of G;

(b) the homomorphism $\alpha: \tilde{G}(k) \cap H \to \tilde{G}'(k')$, where $\tilde{G}'$ is the Zariski closure of $\alpha(\tilde{G}(k) \cap H)$ in G', is representable in the form $\alpha(\tilde{h}) = \tilde{\gamma}(\tilde{h})\,\tilde{\beta}(\tilde{\varphi}^\circ(\tilde{h}))$, $\tilde{h} \in \tilde{G} \cap H$, with a *unique* field homomorphism $\tilde{\varphi}: k \to k'$, a *unique* k'-isomorphism of algebraic k'-groups $\tilde{\beta}: \tilde{G} \to \tilde{G}'$, and a unique group homomorphism $\gamma: H \cap \tilde{G} \to C(\tilde{G}')$;

(c) the homomorphism $\iota \circ \alpha: \tilde{G}(k) \cap H \to \iota(\tilde{G}')(k')$ is representable in the form $\iota \circ \alpha(\tilde{h}) = \bar{\gamma}(\tilde{h})\,\bar{\beta}(\bar{\varphi}^\circ(\tilde{h}))$, $\tilde{h} \in \tilde{G} \cap H$, with a *unique* field homomorphism $\bar{\varphi}: k \to k'$, a *unique* k'-isomorphism of algebraic k'-groups $\bar{\beta}$
${}^{\bar{\varphi}}\tilde{G} \to \iota(\tilde{G}')$, and a unique group homomorphism $\gamma: H \cap \tilde{G} \to C(\iota(\tilde{G}'))$.

PROPOSITION. *Under the above assumptions there exists a unique field homomorphism $\varphi: k \to k'$ and a unique special k'-isogeny $\beta: {}^{\varphi}G \to G'$ such that $\alpha(h) = \beta(\varphi^\circ(h))$ for $h \in H$. Moreover, $\varphi = \tilde{\varphi}$.*

Proof. Let T be a maximal k-torus of G such that $\tilde{G} \subset G(\Sigma_l)$ where Σ_l is the subsystem of long roots of $\Sigma(G, T)$. We denote $T'' = \beta_1({}^{\varphi}T)$. Let T' be the reduced part of $\iota^{-1}(T')$, so that T' is a maximal torus of G'. Since β_1 is an isomorphism, it follows that $\beta_1(\tilde{G}) \subset G''(\Sigma_l'')$ where $\Sigma_l''' \subset \Sigma(G'', T'')$ is the subsystem of long roots. Since ι inverts lengths of roots, it follows that $\tilde{G}' \subset G'(\Sigma_s')$, where $\Sigma_s' \subset \Sigma(G', T')$ is the subsystem of short roots. Since $\iota \mid G'(\Sigma_s') = \mathrm{Fr}$ (Frobenius map), it follows (by unicity in (b) and (c)) that $\bar{\varphi} = \mathrm{Fr} \circ \tilde{\varphi}$. On the other hand because of unicity statements for $\bar{\varphi}$ it follows that $\bar{\varphi} = \varphi_1$. Thus $\varphi_1 = \mathrm{Fr} \circ \tilde{\varphi}$. Consider now $\beta_2 = \beta_1 \circ \mathrm{Fr}: {}^{\tilde{\varphi}}G \to G''$. By the above we have $\alpha_1 = \beta_2 \circ \tilde{\varphi}$. By [3, no. 3.8] there exists a unique k'-isogeny $\beta: {}^{\tilde{\varphi}}G \to G'$ such that $\beta_2 = \iota \circ \beta$. Since β_1 is an isomorphism and ι is a special isogeny, it follows that the isogeny β is also special. Since ι is a monomorphism on the set of points over a field, it follows that $\alpha(h) = \beta(\tilde{\varphi}(h))$ for $h \in H$. The uniqueness is the consequence of the preceding argument.

6.5. Now we can conclude the proof of the main theorem by exhibiting a subgroup $\tilde{G}$. Again let T be an admissible torus of G. Set $\tilde{G} = G(\Sigma_l)$. By Theorem 4.3 we know that $\tilde{G}'$ is again of type A_2, and therefore the results of [5, Sect. 4] are applicable whence the conditions (b) and (c) are satisfied.

APPENDIX: G_2 IN CHARACTERISTIC 2

The group G_2 has pathological behaviour in characteristic 2. Some of it is described below. To compile this appendix the talks with V. Kac, W. van der Kallen, T. A. Springer, and J. Tits were of great help.

Let k be a field of characteristic 2, G a k-group of type G_2, and $\mathscr{O}$ an octave algebra such that $G = \operatorname{Aut} \mathscr{O}$. Let Q be the norm of $\mathscr{O}$ and Tr the trace of $\mathscr{O}$. Let F be the bilinear form associated to Q. Then F is skew-symmetric and non-degenerate. Let $V = \{x \in \mathscr{O} \mid \operatorname{Tr} x = 0\}$. Then $V \supset k$, the center of $\mathscr{O}$ Set $V' = V/k$ and $X = P(V')$, the associated projective space, $X \simeq \mathbf{P}^5$.

Our group G acts on $\mathscr{O}$ and preserves V and k. Therefore it acts on V' and X. When G acts on V we have $G \subset SO(V, Q \mid V)$, the orthogonal group of type B_3. The action of $SO(V, Q \mid V)$ on V preserves k and, therefore, defines a special k-isogeny κ: $SO(V, Q \mid V) \to Sp(V/k, F \mid (V/k))$, the sympletic group (of type C_3) of the image of F in V/k.

Let us look at the Lie algebras. Let $\mathfrak{g} = \operatorname{Lie} G$, $\tilde{\mathfrak{g}} = \operatorname{Lie} SO(V, Q \mid V)$, and $\tilde{\mathfrak{g}}' = \operatorname{Lie} Sp(V/k, F \mid (V/k))$.

The Lie algebra $\mathfrak{g}$ of G is *simple and is isomorphic to the Lie algebra* (Lie $SL(4)$)/center. Explicitly, if e_{ij}, $i \neq j$, is a (part of) Chevalley basis for Lie $SL(4)$ we map

$$e_{12} \to e_b,$$

$$e_{13} \to e_{a+b},$$

$$e_{14} \to e_{3a+2b},$$

$$e_{23} \to e_a,$$

$$e_{24} \to e_{3a+b},$$

$$e_{34} \to e_{2a+b},$$

To establish that e_{12}, $e_{23} e_{34}$ are simple root vectors use the fact that $[e_a, e_{-2a-b}] = 0$ in char 2. In particular, it follows that

(i) Aut Lie G *is of type* C_3 (cf. also below and compare R. Steinberg (*Pacific J. Math.* **11** (1961), 1119–1129)),

(ii) Lie G *has a non-trivial central extension* (cf. van der' Kallen (Lecture Notes in Mathematics No. Springer-Verlag, Berlin/Heidelberg, New York, 356, 1973)).

On the other hand, $\tilde{\mathfrak{g}}$ has ideal $\mathfrak{h}$ of type $A_1 \times A_1 \times A_1$ corresponding to short roots and a "complement" to the ideal is $\mathfrak{a}$ of type A_3 (long roots). The algebra $\tilde{\mathfrak{g}}'$ has an ideal $\mathfrak{h}'$ of type A_3 (short roots) and its "complement" is of type $A_1 \times A_1 \times A_1$ (long roots). The map $d\kappa$ sends short-root subgroups of $\tilde{\mathfrak{g}}$ to zero and is an isomorphism on long-root subgroups. In particular, Ker $d\kappa \leqslant 9$. Since $\mathfrak{g}$ is simple and dim $\mathfrak{g} = 14$, it follows that

(iii) $d\kappa$ is an isomorphism on $\mathfrak{g} \subset \tilde{\mathfrak{g}}$.

Thus $d\kappa$ imbeds $\mathfrak{g}$ as an ideal into $\tilde{\mathfrak{g}}'$ (therefore $Sp(6) \subset \text{Aut}\,\mathfrak{g}$).

Now let us look at the representations. Let $x \in P(V')$ be a line of highest vectors of V'. Then the reduced stabilizer $G_{x,\text{red}}$ of x in G is a parabolic subgroup of G whose root system is $\Sigma^+ \cup \{-b\}$. The orbit Gx of x in X is closed and has dimension 5. Since dim $P(V') = 5$, we have $Gx = P(V')$.

Thus *G acts transitively on* $P(V') \simeq \mathbf{P}^5$.

Let us study G_x more closely. The Lie algebra $\mathfrak{g} = \text{Lie}\,G$ acts on V'. Since $\mathfrak{g} \simeq (\text{Lie}\,SL(4))/(\text{center})$, we have a representation of a Lie algebra of type A_3 in V'. Using the explicit homomorphism Lie $SL(4) \to \mathfrak{g}$ given above and the fact tat V' has highest weight λ_1 for G (cf. 2.2(iii)) we establish that e_{21}, e_{43} act trivially on the highest weight vector. Thus V' as the representation of Lie $SL(4)$ has highest weight corresponding to the middle node of its Dynkin diagram. It follows that the representation of Lie $SL(4)$ on V' corresponds to the natural representation of Lie $SO(6)$ (recall: $D_3 = A_3$), or, the same, to the representation of Lie $SL(4)$ in skew-symmetric matrices (with zero diagonal).

Thus we have:

Lie G_x has dimension 10 *and is spanned by the Lie algebra of the torus and by the roots subalgebras with roots from $\Sigma^+ \cup \{-b\} \cup \{-2a-b\}$. In particular, G_x is not reduced* (*since* Lie $G_{x,\text{red}} \neq$ Lie G_x).

There is a seeming contradiction: $SL(4)$ has two olbits on $P(V')$: an open one with the stability group of type B_2 and a closed one with a parabolic stability group. Their Lie algebras are not conjugate by $SL(4)$. However, since G acts transitively on $P(V')$, it follows that the corresponding Lie subalgebras in $\mathfrak{g}$ are conjugate by G. Recall in this connection that images of these subalgebras in $\mathfrak{g}$ are solvable of the same dimension.

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