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Abstract Isomorphisms of Simple Algebraic Groups Split by Quadratic Extensions

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Abstract isomorphisms of a big subgroup of an anisotropic absolutely almost simple algebraic group split by a separable quadratic extension of the base field with big subgroups of other semi-simple algebraic groups are shown to have a standard decomposition into a field isomorphism, a special isogeny of algebraic groups and a radial isomorphism.

0. INTRODUCTION

0.1. The present paper concludes a series of papers [7, 20–23]. Its original version was written several years ago. But at that time I was not able to handle the case of groups of type B_2 in characteristic 2 and the case of groups of type G_2. This created a gap in the result, which was filled by [21, 22]. Reference [23] was originally an attempt to search for new technique but, unexpectedly, it also led to a new type of results.

The origin of the present paper was in an attempt to prove an analog of the theorem of A. Borel and J. Tits [4] for groups split over quadratic extensions. This goal was not achieved: arbitrary homomorphisms or even monomorphisms are outside my reach.

The method of the present paper is inspired by O. T. O'Meara's method of residual spaces: it so happened that I saw papers [4] and [5] at the same time, and I tried to fit them in a common picture.

0.2. The main result (cf. Section 6) contains as a particular case the following assertion (where we use notation of [4]):

THEOREM. *Let k and k' be infinite fields; let K be a quadratic separable extension of k. Let G be an anisotropic absolutely almost simple connected*

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and simply connected algebraic k-group of rank $\geqslant 2$ *split by K and let* G' *be an absolutely almost simple connected and simply connected* k'*-group.*

Suppose that we have an abstract isomorphism $\alpha: G(k) \to G'(k')$. *Then there exist a unique field isomorphism* $\varphi: k \to k'$, *a unique special* k'*-isigeny* $\beta: {}^{\varphi}G \to G'$, *and a unique homomorphism* $\gamma: G(k) \to \mathrm{Center}(G'(k'))$ *such that*

$$\alpha(h) = \gamma(h) \cdot \beta(\varphi^0(h)) \qquad \textit{for } h \in G(k).$$

Moreover, β *is an isomorphism unless G is of type* G_2 *and* char $k = 3$.

Here ${}^{\varphi}G$ is the k'-group obtained from G by the base change φ and $\varphi^0: G(k) \to {}^{\varphi}G(\varphi(k))$ is the corresponding isomorphism.

0.3. Let me briefly describe the relation of the present paper to the known results. There are for me three high points in the theory of abstract homomorphisms between algebraic groups:

(a) the result of A. Borel and J. Tits [4];

(b) the results of O. T. O'Meara (e.g., [12, 13]) and his school (E. A. Connors, A. J. Hahn, A. A. Johnson, R. Solazzi, *et al.*) which continued the development begun by Schreier and van der Waerden and followed up by J. Dieudonné and Hua Lo Keng;

(c) the result of G. Margulis [10], G. Mostow, and G. Prasad.

The setting for all these results is roughly as follows. (Roughly in the sense that what is said below is not completely precise but only outlines the picture.) Let k and k' be infinite fields, let G (resp., G') be an absolutely almost simple algebraic k- (resp., k'-) group, let H be a subgroup of $G(k)$, and let $\alpha: H \to G'(k')$ be a homomorphism with dense image. Under certain assumptions on α, G, H, G', and the image $\alpha(H) \subseteq G'(k')$ one derives that there exists a unique field homomorphism $\alpha: k \to k'$, a unique special k'-isogeny $\beta: {}^{\varphi}G \to G'$, and a unique homomorphism $\gamma: H \to C(G'(k'))$ such that

$$\alpha(h) = \gamma(h) \cdot \beta(\varphi^0(h)) \quad \text{for all } h \in H.$$

The assumptions of Borel and Tits are (i) G is isotropic over k and (ii) H contains all "good" unipotents from $G(k)$. The assumptions of O'Meara's school are (i) G and G' are classical (SL_n, Sp_{2n}, SO_n, SU_n, and isogeneous groups), (ii) H and $\alpha(H)$ are "full" in G and G', respectively (the notion of a full group includes certain integral subgroups and their congruence subgroups), (iii) α is an isomorphism. My assumptions are (i) G is anisotropic over k but split by a quadratic separable extension of k and $rkG \geqslant 2$, (ii) H and $\alpha(H)$ are "big" subgroups of G and G', respectively, and (iii) α is an isomorphism.

The results of group (c) are much stronger than the results of groups (a) and (b) but they are proved in very special situations. So we skip a comparison with them.

The above shows that the results of the present paper are somewhere in between the results of groups (a) and (b).

We refer to [4, 6, 13, 17, 18] for a historical survey and to [11, pp. 255–259] for a rather complete bibliography of papers on homomorphisms between algebraic groups covering the years 1928–1978.

0.4. *Conventions and Notation*

Let k be a field. Then k_s (resp., $\bar{k}$) denotes separable (resp., algebraic) closure of k. If G is an algebraic group defined over k, then $G(k)$ denotes the set of rational points of G. If $\varphi: k \to k'$ is a homomorphism of fields, then ${}^{\varphi}G$ denotes the algebraic $\varphi(k)$-group obtained from G by the base change φ, and $\varphi^0: G(k) \to {}^{\varphi}G(k')$ denotes the corresponding homomorphism of points.

By an algebraic group we understand a reduced affine group scheme of finite type over a field. We denote by G^0 the connected component of G. A reductive group is always assumed to be connected. For a torus T we denote by $X(T)$ its group of characters, and, if T is defined over k, then $\Gamma(T)$ denotes the Galois group of the minimal splitting field of T.

If T is a subtorus of an algebraic group G, then $\Sigma(G, T)$ denotes the set of roots of T in G (or, the same, in the Lie algebra of G). Let Σ_1 be a subset of $\Sigma(G, T)$. Then $\mathbf{Q}\Sigma_1$ denotes the subspace of the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes X(T)$ generated by Σ_1. We say that Σ_1 is saturated if $\Sigma_1 = \mathbf{Q}\Sigma_1 \cap \Sigma(G, T)$ (the intersection is taken inside $\mathbf{Q} \otimes X(T)$). Next, $G(\Sigma_1)$ denotes the subgroup of G generated by the root subgroups of T in G corresponding to roots from $\pm\Sigma_1$. In particular, if T is a maximal torus, G is reductive and $a \in \Sigma(G, T)$, then $G(a)$ is a three-dimensional semi-simple subgroup of G. For a subset $D \subseteq T$ we set $\Sigma(D) = \{a \in \Sigma(G, T) | a(D) = 1\}$ and for a subset $\tilde{\Sigma} \subset \Sigma(G, T)$ we set $D(\tilde{\Sigma}) = \{d \in D | a(d) = 1 \text{ for all } a \in \tilde{\Sigma}\}$. If T is a maximal torus and G is semi-simple, then for $a \in \Sigma(G, T)$ we denote by U_a the root subgroup corresponding to a and $x_a(t)$ its standard parametrization (so that $x_a: \mathbf{G}_a \simeq U_a$). The Weyl group of $\Sigma(G, T)$ is denoted $W(\Sigma(G, T))$.

A pair of reductive algebraic groups G, G' have the same root data if there exists an isomorphism $\beta: G \to G'$ such that for a maximal torus T of G the derived mapping $\beta^*: X(\beta(T)) \to X(T)$ induces an isomorphism $\beta^*: \Sigma(G', \beta(T)) \simeq \Sigma(G, T)$.

For an algebraic (resp., abstract) group G and an algebraic (resp., abstract) subset M of G we denote by $Z_G(M)$, $N_G(M)$, $\mathscr{D}^i(G)$, and $C(G)$ the algebraic (resp. abstract) subgroup of G which is respectively the centralizer of M in G, the normalizer of M in G, the ith derived group of G, and the center of G. If G is as above and $M_1,..., M_n$ are algebraic (resp., abstract) subsets of G, then $\langle M_1,..., M_n \rangle$ denotes the algebraic (resp., abstract) group generated by the groups $M_1,..., M_n$.

The prefixed letter P stands for projectivisation. Explicitly, if V is a vector space, then PV denotes the corresponding projective space. If $\alpha: V \to V'$ is an imbedding of vector spaces, then $P\alpha: PV \to PV'$ is the induced mapping of projective spaces. If G is a group, then $PG = G/G(G)$; if $\alpha: G \to G'$ is a group homomorphism such that $\alpha(C(G)) \subseteq C(G')$, then $P\alpha: G/C(G) \to G'/C(G')$.

Finally, $|X|$ denotes the cardinality of X.

1. Recollections about Groups Split over Separable Quadratic Extensions

We state below some results of [19] and their easy corollaries.

Let k be a field and let K be its quadratic separable extension. Set $\mathrm{Gal}(K/k) = \{1, \sigma\}$. An absolutely almost simple anisotropic k-group which is split over K is called *K-admissible*. A maximal k-torus of G is called *K-admissible* if it is split over K.

1.1. Pproposition (cf. [19, nos. 3, 4]). *Let G be K-admissible.*

(i) *G contains K-admissible tori;*

(ii) *if T is a K-admissible torus of G, then*

$$\sigma a = -a \qquad \text{for } a \in X(T);$$

(iii) *the groups $G(a)$, $a \in \Sigma(G, T)$, are defined over k.*

1.2. We have a partial converse:

Proposition. *Let H be a semi-simple k-group and let S be a maximal k-subtorus of H. If the groups $G(a)$, $a \in \Sigma(H, S)$, are defined over k, then T is either split by k or anisotropic and split by a quadratic separable extension of k.*

Proof. The assumptions imply that $wa = \pm a$ for $w \in \Gamma(S)$, $a \in \Sigma(H, S)$. The only elements of $\mathrm{Aut}\, \Sigma(H, S)$ which have this property are ± 1. Thus $\Gamma(S)$ acts as a subgroup of ± 1 on Σ. If this subgroup is $\{1\}$, then T is split over k. If this subgroup is $\{\pm 1\}$ itself, we have the second case.

1.3. Let G' be another K-admissible group and let T' be a K-admissible subtorus of G'. Let $\tilde{\alpha}: X(T) \to X(T')$ be an isomorphism such that $\tilde{\alpha}(\Sigma(G, T)) = \Sigma(G', T')$ (i.e., $\tilde{\alpha}$ is an isomorphism of root data). Let Δ be a system of simple roots of Σ. Suppose that for all $a \in \Delta$ we are given a k-isomorphism $\beta_a: G(a) \to G'(\tilde{\alpha}(a))$ such that $\beta_a(T \cap G(a)) = T' \cap G'(\tilde{\alpha}(a))$ and $\beta_a(x_a(t)) = x'_{\tilde{\alpha}(a)}(t')$.

PROPOSITION. *Suppose, in addition, that G is simply connected or adjoint if G is of type $D_{2n}, n > 1$. Then there exists a unique k-isomorphism $\beta: G \to G'$ such that $\beta | G(a) = \beta_a | G(a)$ for $a \in \Delta$.*

Proof. This is standard and is contained, for example, in [23, 8.7]. This latter result is applicable since for groups of type not D_{2n}, $n > 1$, there exists a unique central isogeny of a simply connected group onto our group.

1.4. Two K-admissible tori T and T_1 of G are called *associated* with respect to $a \in \Sigma(G, T)$ if $T_1 \subseteq T \cdot G(a)$. We know (by [19, no. 7]) that any pair of K-admissible tori of G can be connected by a sequence of length $\leqslant |\Sigma^+|$ of associated tori. We will need a slightly sharper version of this result.

1.4.1. Let B be a Borel K-subgroup, containing T; let $w_0 = s_{a_1} \cdots s_{a_d}$ be a reduced decomposition of the opposition symmetry $w_0 \in W(\Sigma)$; let $P_i = G(a_i) B$.

LEMMA ([23, Lemma 3.3]). $G(K) = P_1(K) \cdot P_2(K) \cdots P_d(K)$.

1.4.2. Let T' be another K-admissible qubtorus of G and let B' be a Borel K-subgroup containing T'. Then there exists $g \in G(K)$ such that $gTg^{-1} = T'$, $gBg^{-1} = B'$. Applying 1.4.1 to the expression $w_0 = w_0^{-1} = s_{a_d} s_{a_{d-1}} \cdots s_{a_1}$ we find $p_i \in P_i(K)$ such that $g = p_d \cdots p_1$. We set $g_0 = 1$, $g_i = p_i \cdots p_1$, $B_0 = B$, $B_i = g_i B g_i^{-1}$, $T_i = B_i \cap B_i^{\sigma}$, $\tilde{P}_i = g_{i-1} P_i g_{i-1}^{-1}$, $G_i = \tilde{P}_i \cap \tilde{P}_i^{\sigma}$.

LEMMA. *The sequence of tori $T_0 = T$, $T_1, \ldots, T_d = T'$ consists of K-admissible tori and T_{i+1} is associated to T_i with respect to some $b_i \in \Sigma(G, T_i)$.*

Proof is essentially contained in [19, no. 7].

1.4.3. *Remark.* It was shown in [7] that the association theorem holds also for isotropic unitary groups but only with a greater number of associations.

6. BIG SUBGROUPS

We define below the class of groups which will be our main object. We start with a standard lemma.

2.1. LEMMA. *Let G be a connected algebraic k-group and $G_1, G_2, \ldots, G_m$ closed algebraic k-subgroups of G. Let H be a dense subgroup of G and let H_i be a dense subgroup of G_i.*

(i) *If* H_1 *is normal in* H, *then* G_1 *is normal in* G *and* $H/H \cap G_1$ *is dense in* G/G_1;

(ii) $Z_G(H_1) = Z_G(G_1)$ *and* $Z_H(H_1) = Z_G(G_1) \cap H$;

(iii) $\mathscr{D}^i H$ *is dense in* $\mathscr{D}^i G$;

(iv) *if* G *is connected and reductive, then* $\mathscr{D}^i H$ *is dense in* $\mathscr{D}G$ *for all* $i \geqslant 1$; *in particular, if* G *is not a torus, then* $\mathscr{D}^i H \neq 1$ *for* $i \geqslant 1$;

(v) *if* G_1 *is connected and reductive, then* $Z_H(\mathscr{D}^i H_1) = Z_H(\mathscr{D}H_1) = H \cap Z_G(\mathscr{D}G_1)$;

(vi) $\langle H_1, ..., H_m \rangle$ *is dense in* $\langle G_1, ..., G_m \rangle$;

(vii) *if* G *is not unipotent,* H *contains a regular element of* G *of arbitrary large order.*

Proof. In (i) it is clear that G_1 is normal in G. Let $\bar{G}$ be the closure of HG_1/G_1 in G/G_1. Then the preimage of $\bar{G}$ contains H whence $\bar{G} = G/G_1$. Assertion (ii) is clear. Now (iii) follows from (i) if we take G_1 to be the closure of $H_1 = \mathscr{D}H$. Next, (iv) holds because $\mathscr{D}G = \mathscr{D}^i G$ for $i > 1$ and because of (iii). Now (v) is the combination of (ii) and (iv). To prove (vi) note that $H_{i_1} \times H_{i_2} \times \cdots \times H_{i_n}$ is dense in $G_{i_1} \times G_{i_2} \times \cdots \times G_{i_n}$ whence the image of the first product in G is dense in the image of the second. Finally, (vii) holds since the set of regular elements of order $\geqslant n_0$ is open in G and non-empty for any $n_0 \in \mathbf{Z}$.

2.2. Let k be an infinite field and let T be a k-torus.

DEFINITION. A dense subgroup M of $T(k)$ is called *big* if for every two k-subtori T_1, T_2 of T, the group $M/(M \cap T_1)(M \cap T_2)$ is periodic if and only if $T = T_1 T_2$.

To handle big subgroups we need

2.2.1. LEMMA. *Let* T *and* T' *be two* k*-tori and let* $\beta: T \to T'$ *be a surjective* k*-homomorphism. Then* $T'(k)/\beta(T(k))$ *is periodic.*

Proof. Let $\tilde{T} = (\operatorname{Ker} \beta)^0$. Let $\tilde{\beta}: \tilde{T} \to T$. Then $\tilde{\beta}^*: X(T) \to X(\tilde{T})$ and β^* is surjective. Let $\tilde{Y} = \operatorname{Ker} \tilde{\beta}^*$ and let Z be a $\Gamma(T)$-complement to $\mathbf{Q}\tilde{Y}$ in $\mathbf{Q}X(T)$. (Since $\Gamma(T)$ is finite, its representation on $\mathbf{Q}X(T)$ is completely reducible and therefore Z exists.) Let π be the projection of $\mathbf{Q}X(T)$ onto Z and let $\bar{T}$ be the k-torus whose chracter group is $\pi(X(T))$. Then we have a surjective k-map $\bar{\beta}: \tilde{T} \times \bar{T} \to T$. Thus $\beta \circ \bar{\beta}: \bar{T} \to T'$ is a k-isogeny and therefore by [4, 3.17] $T'(k)/\beta \circ \bar{\beta}(\bar{T}(k))$ is periodic. Since $\bar{\beta}(\bar{T}(k)) \subseteq T(k)$, it follows that $T'(k)/\beta(T(k))$ is periodic.

2.2.2. LEMMA. (i) *$M = T(k)$ is a big subgroup of $T(k)$.*

(ii) *Let m be an integer, let M be a big subgroup of $T(k)$, and let $N = \langle h^m, h \in M\rangle$. Then any subgroup $\tilde{M}$ of $T(k)$ such that $M \supseteq \tilde{M} \supseteq N$ is a big subgroup of $T(k)$.*

(iii) *Let $\beta: T \to T'$ be a surjective k-homomorphism and let M be a big subgroup of $T(k)$. Then $\beta(M)$ is a big subgroup of $T'(k)$.*

Proof. Let us start with (i). If $T \neq T_1 T_2$, then $T/T_1 T_2$ is a non-trivial k-torus. Since $T(k)/T_1(k)\, T_2(k)$ is dense in $T/T_1 T_2$ (by Lemma 2.1(i)) and k is infinite, it follows that $T(k)/T_1(k)\, T_2(k)$ is not periodic. If $T = T_1 T_2$, then we apply Lemma 2.2.1 to the map $T_1 \times T_2 \to T$ and get the validity of (i).

To prove (ii) we remark first that $(M \cap T_1)/(N \cap T_1)$ and, therefore, $(M \cap T_1)/(\tilde{M} \cap T_1)$ are periodic (of period m) for any k-subtorus T_1 of T. Let T_1, T_2 be two k-subtori of T. We have $\tilde{M}/(\tilde{M} \cap T_1)(\tilde{M} \cap T_2)$ is periodic iff $M/(\tilde{M} \cap T_1)(\tilde{M} \cap T_2)$ is periodic and the latter is periodic iff $M/(M \cap T_1)(M \cap T_2)$ is periodic, i.e., since M is big, iff $T = T_1 T_2$. Thus $\tilde{M}$ is big. Now let us prove (iii). For two k-subtori T'_1, T'_2 of T' we set $T_i = (\beta^{-1}(T'_i))^0$. Then the map $\beta: T_i \to T'_i$ is surjective, and $T'_1 T'_2 = T'$ if and only if $T_1 T_2 = T$. By Lemma 2.2.1 the groups $T'_i(k)/\beta(T_i(k))$, $i = 1, 2$, are periodic and so is $T'(k)/\beta(T(k))$. It follows that the groups $(\beta(M) \cap T'_i)/\beta(M \cap T_i)$, $i = 1, 2$, are periodic, and therefore $\beta(M)/(\beta(M) \cap T'_1)(\beta(M) \cap T'_2)$ is periodic iff $\beta(M)/\beta(M \cap T_1)(\beta(M \cap T_2)$ is periodic. But this latter group is periodic iff $T = T_1 T_2$. Thus the former group is periodic iff $T' = T'_1 T'_2$. Thus $\beta(M)$ is big.

2.2.3. LEMMA. *Let M be a big subgroup of $T(k)$. Then for every k-subtorus T_1 of T the group $M \cap T_1$ is dense in T_1.*

Proof. Let $X = X(T)$, $\Gamma = T(T)$, $X_1 = \{a \in X \mid a(T_1) \equiv 1\}$. Let $\mathbf{Q}X_2$ be a Γ-stable complement to ${}_{\mathbf{Q}}X_1$ in $\mathbf{Q}X$. Let $X_2 = \mathbf{Q}X_2 \cap X$ and let T_2 be the connected component of $\{t \in T \mid X_2(t) = 1\}$. Then T_2 is a k-subtorus, $T_1 \cap T_2$ is finite, and $T = T_1 T_2$. In particular, $M/(M \cap T_1)(M \cap T_2)$ is periodic whence $(M \cap T_1)(M \cap T_2)$ is dense in T. Since $T_1 \cap T_2$ is finite, it follows that $M \cap T_i$ is dense in T_i, $i = 1, 2$, as required.

2.3. Let k be an infinite field and let H be a reductive algebraic k-subgroup.

2.3.1. DEFINITION. A subgroup $M \subseteq H(k)$ is called *big* if $M \cap T$ is a big subgroup of $T(k)$ for every maximal k-torus T of H.

2.3.2. LEMMA. (i) *$H(k)$ is a big subgroup of $H(k)$.*

(ii) *If $\beta: H \to H'$ is a k-homomorphism of reductive k-groups such that* *the (schematic) kernel of β belongs to the center of H, then the image of a big subgroup of $H(k)$ under β is a big subgroup of $H'(k)$.*

(iii) *Let m be an integer and let M be a big subgroup of $H(k)$. Then the group $\langle h^m, h \in M\rangle$ is a big subgroup of $H(k)$.*

Proof. We have: (i) follows from Lemma 2.2.2(i); (ii) follows from Lemma 2.2.2(iii) and [3, 2.11(iv), 2.17]; (iii) follows from Lemma 2.2.2(ii).

2.3.3. LEMMA. (i) *Let $\tilde{H}$ be a reductive k-subgroup of H and let M be a big subgroup of $H(k)$. Then $M \cap \tilde{H}$ is dense in $\tilde{H}$.*

(ii) *If $\tilde{H}$ in* (i) *contains a maximal k-subtorus of H, then $M \cap \tilde{H}$ is big.*

Proof. Statement (ii) is evident. Let us prove (i). By [1, 7.10, 7.12] the set of maximal k-tori of $\tilde{H}$ is dense in the set of tori of $\tilde{H}$. If $\tilde{T}$ is a maximal k-torus of H, then T is contained (again by [1, 7.10]) in a maximal k-torus T of H. By Lemma 2.2.3 we know that $M \cap \tilde{T}$ is dense in $\tilde{T}$. Therefore $M \cap \tilde{H}$ is dense in $\tilde{H}$.

2.3.4. PROPOSITION. *Let M be a big subgroup of $H(k)$ and let T be a k-subtorus of H. Then the following statements are equivalent:*

(i) $T = C(Z_H(T)^0)$,

(ii) $T = C(Z_H(T \cap M)^0)$,

(iii) $C(Z_M(T \cap M))/T \cap M$ *is periodic.*

Proof. Since M is big, we have $Z_H(T) = Z_H(T \cap M)$ by Lemma 2.1(ii) and 2.2.3. This establishes (i)$\sim$(ii). Now (iii) implies that T has a finite index in $C(Z_H(T \cap M))$, i.e., it implies (ii). If T has a finite index m in $C(Z_H(T \cap M))$, then $[C(Z_H(T \cap M))(k)]^m \subseteq T(k)$ whence $[C(Z_M(T \cap M))]^m \subseteq T(k) \cap M$. So (ii) implies (iii).

2.3.5. LEMMA. *Let M be a big subgroup of H, let P be a parabolic k-subgroup of H, and let U be the unipotent radical of P. Then $M \cap U$ is dense in U.*

Proof. Let R be the radical of P. Then the set of maximal k-tori of R is dense in the set of tori of R. It follows from this together with Lemmas 2.3.3 and 2.1(iii) that $[R \cap M, R \cap M]$ is dense in $U = [R, R]$, as required.

2.4. Now let k and k' be infinite fields. Let H and H' be reductive algebraic k- and k'-groups. Let M be a big subgroup of H and let M' be a big subgroup of H'. Let $\alpha: M \to M'$ be a group isomorphism.

2.4.1. LEMMA. *The group $\mathscr{D}H$ is isotropic over k if and only if the group $\mathscr{D}H'$ is isotropic over k'.*

Proof. Suppose that $\mathscr{D}H$ is isotropic. Let P be a minimal parabolic k-subgroup of H. So $P \neq H$. Let R be the radical of P. Then (as in Lemma 2.3.5) $R \cap M$ is dense in R. Now the argument of the proof of [4, 7.1(i)] concludes the proof.

2.4.2. *Remark.* The above lemma justifies our later assumption that our groups are anisotropic.

2.4.3. LEMMA. *Let T be a maximal k-subtorus of H. Then there exists a semi-simple element $h \in T \cap M$ such that $\alpha(h)$ is a semi-simple regular element of H' and $Z_{M'}(\alpha(h)) = Z_{H'}(\alpha(h))^0 \cap M'$.*

Proof. Let p' be the characteristic exponent of k' and let $m = p^{\dim G'}$. Take a strongly regular (in the sense of [16]) element $h \in M \cap T$ such h^m is also strongly regular. Let us show first that $\alpha(h^m)$ is semi-simple. Let $\alpha(h) = t \cdot n$ be the Jordan decomposition (in G') with t semi-simple and n unipotent, $tn = nt$. Suppose that $n \neq 1$. If $p' \neq 1$, then $n^m = 1$ whence $(h^m) = t^m n^m = t^m$ is semi-simple. If $p' = 1$, we imbed n into a connected one-dimensional unipotent k-subgroup N of H' normalized by a connected one-dimensional k-torus S (Jacobson–Morozov theorem). The argument of the proof of Lemma 2.3.5 shows the group $\alpha^{-1}(M' \cap N)$ contains unipotents. On the other hand we have $Z_{H'}(n_1) = Z_{H'}(N)$ for any $n_1 \in N$, $n_1 \neq 1$. In particular, $\alpha(h)$ commutes with N whence h commutes with $\alpha^{-1}(M' \cap N)$. But we have $Z_H(h) = T$, a torus, which is a contradiction. Now let us show that $\alpha(h^m)$ is regular. If it is singular, then $H'_1 = Z_{H'}(\alpha(h^m))^0$ is a reductive non-commutative k'-group. By Lemmas 2.3.3 and 2.1(iv) the group $Z_{M'}(\alpha(h^m))$ is non-commutative. This is a contradiction since h^m is strongly regular and $Z_H(h^m) = T$ whence $Z_{M'}(\alpha(h^m)) \subseteq \alpha(M \cap T)$. Now let us show that h itself is semi-simple. We know that $T' = Z_{H'}(\alpha(h^m))^0$ is a torus. It is defined over k' since $\alpha(h^m) \in H'(k')$ and by [2, 10.3]. Since $T' \subset Z_{H'}(\alpha(h^m))$ we see that $\alpha^{-1}(T' \cap M) \subset T$, whence it follows that $\alpha(h)$ commutes with $T' \cap M'$, whence by Lemma 2.3.3 it follows that $\alpha(h) \in T'$.

Thus we have that $\alpha(h)$ is semi-simple regular. It remains to establish that $Z_{M'}(\alpha(h)) = T' \cap M'$. Suppose $\tilde{h} \in Z_{M'}(\alpha(h))$, $\tilde{h} \notin T'$. Then $\tilde{h} \in N_{H'}(T')$ and therefore acts non-trivially on T'. Therefore $\alpha^{-1}(\tilde{h})$ acts non-trivially on $\alpha^{-1}(T' \cap M)$. This is a contradiction.

2.4.4. PROPOSITION. *Let $\tilde{h} \in M$. Then $\alpha(\tilde{h})$ is semi-simple if and only if $\tilde{h}$ is.*

Proof. Since the situation is invertible, it is enough to prove that $\tilde{h}$ semi-simple implies that $\alpha(\tilde{h})$ also is semi-simple. Let T be a maximal k-torus containing $\tilde{h}$. Then T contains an element h of Lemma 2.4.3. We have $h\tilde{h} = \tilde{h}h$ whence $\alpha(\tilde{h}) \in Z_{H'}(\alpha(h)) \cap M'$. Since $Z_{H'}(\alpha(h)) \cap M' \subseteq Z_{H'}(\alpha(h))^0$, our assertion follows.

2.4.5. COROLLARY. *α establishes a bijection between the sets of maximal k- and k'-tori of H and H'. We denote this bijection α_*. It is characterized by the property $\alpha_*(T) \cap M' = \alpha(T \cap M)$.*

2.4.6. PROPOSITION. *Let $\tilde{H}$ be a reductive k-subgroup of H of maximal rank. Then the connected component of the Zariski closure $\tilde{H}'$ of $\alpha(\tilde{H} \cap M)$ is a reductive k'-subgroup of H'.*

Proof. Let T be a maximal k-subtorus of $\tilde{H}$. Then $T' = \alpha_*(T)$ is a maximal k'-torus of H'. Therefore $\tilde{H}'$ is normalized by a maximal k'-torus and therefore $\tilde{H}'$ and $\tilde{H}'^0$ are defined over k'. If $\tilde{H}'^0$ is not reductive, then its unipontent radical is defined over k' (it follows, for example, from [1, 8.2]) and the argument of the proof of Lemma 2.3.5 would show that $\tilde{H}$ is not reductive, a contradiction. So $\tilde{H}'^0$ is reductive.

2.4.7. COROLLARY. *Let $\tilde{H}$ be a reductive k-subgroup of H normalized by a maximal k-torus T of H. Then the connected component $\tilde{H}'^0$ of the Zariski closure $\tilde{H}'$ of $\alpha(\tilde{H} \cap M)$ in H' is a reductive k'-group. Moreover, $\tilde{H}'^0$ is semi-simple if $\tilde{H}$ is.*

Proof. Apply Proposition 2.4.6 to $\tilde{H} \cdot T$. It gives us all assertions except the last one. Let $n = [\tilde{H}' : \tilde{H}'^0]$. Then $N = \langle m^n, m \in M \cap \tilde{H} \rangle$ is a dense subgroup of $\tilde{H}$ and $\alpha(N) \subset \tilde{H}'^0$. Replacing M by N we obtain that $\tilde{H}' = H'^0$. Now $C(H)$ is infinite if and only if $C(\tilde{H}'^0)$ is whence the last assertion.

2.4.8. LEMMA. *Suppose that $H/C(H)$ is k-simple. Then $H'/C(H')$ is k'-simple.*

Proof. Let $H'/C(H') = H_1' H_2'$ be a direct product of k'-simple groups. By Lemma 2.3.5 we have $M' \cap H_i'$ is dense in H_i'. Then $\alpha^{-1}(M' \cap H_i')$ are infinite commuting normal subgroups of M. The connected components H_1, H_2 of their Zariski closures are non-trivial normal subgroups of $H/C(H)$. By Corollary 2.4.7, H_1 and H_2 are defined over k, a contradiction with the assumptions.

2.4.9. COROLLARY. *Suppose that $H/C(H)$ is k-simple. Then there exists a finite separable extension k'' of k', an absolutely simple k''-group H'', and a big subgroup M'' of H'' such that $H'/C(H') = R_{k''/k'}(H'')$ and $M'/C(M') = R_{k''/k'}(M'')$.*

3. SUBTORI AND ROOT SETS

We define some sets of k-subtori in a reductive k-group. These sets of subtori determine some kind of geometry connected with our group. Then we show that these sets of subtori are "preserved" by an isomorphism of big subgroups of two algebraic reductive k-groups. This permits one to prove that the two algebraic groups are very similar (it will be done in the next section). To make use of our sets of tori we identify them with some sets of root subsystems of our original group.

3.1. Let k be an infinite field, let H be a reductive algebraic k-group, let T be a k-subtorus of H, let $\Gamma = \Gamma(T)$, and let M be a big subgroup of H.

Assume that $T = C(Z_H(T)^0)$. Let $\Sigma_T = \Sigma(H, T)$ be the set of roots of T in H.

3.1.1. DEFINITION OF $\tilde{\Pi}_i(H, T)$. The set $\tilde{\Pi}_i(H, T)$ is a set of k-subtori of T which is defined inductively by (a) $\tilde{\Pi}_1(H, T) = \{T\}$, and (b) $\tilde{\Pi}_{i+1}(H, T)$ is the set of k-subtori $T' \subseteq T$ such that

(i) $T' = C(Z_H(T')^0)$,

(ii) there exists $T'' \in \tilde{\Pi}_i(H, T)$ such that $T'' \supset T'$, $T'' \neq T'$ and for every k-subtorus T''' such that $T'' \supset T''' \supset T'$ and $T''' \neq T'$ one has $C(Z_H(T''')^0) = T''$ (i.e., T' is maximal in T'' among k-tori having property (i)).

3.1.2. DEFINITION OF $\tilde{\Pi}_i(M, T)$. The set $\tilde{\Pi}_i(M, T)$ is a set of subgroups of $M \cap T$ which is defined inductively by

(a) $\tilde{\Pi}_1(M, T) = \{T \cap M\}$;

(b) $\tilde{\Pi}_{i+1}(M, T)$ is the set of subgroups $D' \subseteq T \cap M$ such that

(i) $D' = C(Z_M(D'))$,

(ii) there exists $D'' \in \tilde{\Pi}_i(M, T)$ such that $D'' \supset D'$, D''/D' is not periodic and for any $t \in D''/D'$ of infinite order and any $t_1 \in tD'$ one has $C(Z_M(t_1, D')) = D''$.

(iii) D' is minimal with properties (i) and (ii).

Remark. The last condition is a substitute for connected component.

3.1.3. DEFINITION OF $\Pi_i(H, T)$. The set $\Pi_i(H, T)$ is a set of saturated Γ-stable subsets of Σ_T which is defined inductively by

(a) $\Pi_1(H, T) = \{\phi\}$,

(b) $\Pi_{i+1}(H, T)$ consists of saturated Γ-stable subsets of Σ_T such that there exists $\Sigma'' \in \Pi_i(H, T)$ such that $\Sigma'' \subset \Sigma'$ and Σ'' is a maximal Γ-stable saturated subset of Σ'.

3.1.4. Let us define a mapping $\tilde{\pi}_i: \tilde{\Pi}_i(M, T) \to \tilde{\Pi}_i(H, T)$ by $\tilde{\pi}_i(D) = C(Z_H(D)^0)$ and a mapping $\tilde{\pi}'_i: \tilde{\Pi}_i(H, T) \to \tilde{\Pi}_i(M, T)$ by $\tilde{\pi}'_i(T') = C(Z_M(M \cap T))$.

PROPOSITION. *The above mappings*

(i) *are well defined,*

(ii) *are inverse to one another.*

In particular, they establish an identification of $\tilde{\Pi}_i(H, T)$ *with* $\tilde{\Pi}_i(M, T)$.

Proof. The above statements are true if $i = 1$. Let us assume that they are true for $i \leqslant m$ and let us prove by induction that they are true also for $i = m + 1$.

If $T' \in \tilde{\Pi}_{m+1}(H, T)$, then $T' = C(Z_H(T'))^0$. By Lemma 2.3.4 we have $Z_H(T' \cap M) = Z_H(T')$. Let $D' = C(Z_M(T' \cap M))$ $(=\pi'_{m+1}(T'))$. Then $T' \cap M \subseteq D'$ and $Z_H(T') = Z_H(D')$, i.e., Definition 3.1.2(i) holds. Now take $T'' \in \tilde{\Pi}_m(H, T)$ as in Definition 3.1.1(ii). Then by the inductive assumption $D'' = \tilde{\pi}'_m(T'') \in \tilde{\Pi}_m(M, T)$. Let us show that Definition 3.1.2(ii) holds with D'' chosen as above. Suppose that it does not hold. Take then t, t_1 as in Definition 3.1.2(ii). We have $C(Z_M(D', t_1)) \neq D'$. Let $T''' = C(Z_H(D', t_1))^0$. Since t is of infinite order, we have $T' \neq T'''$. Therefore by Definition 3.1.1(ii) we have $C(Z_H(T'''))^0 = T''$. Set $D''' = C(Z_M(D', t_1))$. Then the above means that $\tilde{\pi}_m^{-1}(T'') = D'''$ and the inductive assumptions imply that $D'' = \tilde{\pi}'_m(T'') = D'''$, a contradiction with the assumption $D'' \neq D'''$. It remains to check that Definition 3.1.2(iii) holds for D'. Again suppose that it does not hold. Then $D' \supset \tilde{D}'$ such that $\tilde{D}'$ has properties 3.1.2(i) and 3.1.2(ii) and $D' \neq \tilde{D}'$. Let $\tilde{T}' = C(Z_H(\tilde{D}'))^0$. Then we must have (by Definition 3.1.2(ii)) that $\tilde{T}' = T'$ whence $\tilde{D}' = C(Z_M(T' \cap M)) = D'$ as required. Thus $\tilde{\pi}'_{m+1}$ is well defined.

Now let $D' \in \tilde{\Pi}_{m+1}(M, T)$. Then $D' = C(Z_M(D'))$. Set $T' = C(Z_H(D'))^0$ $(= \tilde{\pi}_{m+1}(D'))$. We have, therefore, $T' = C(Z_H(T'))^0$, i.e., T has property 3.1.1(i). Take $D'' \in \tilde{\Pi}_m(M, T)$ as in Definition 3.1.2(ii). Then the induction assumptions say that $T'' = \tilde{\pi}'_m(D'')$ belongs to $\tilde{\Pi}_m(H, T)$. Let T''' be a k-subtorus of T such that $T'' \supseteq T''' \supseteq T'$, $T''' \neq T'$. Then $M \cap T'''/\cap T'$ is not periodic. Take $t \in (M \cap T''')/(M \cap T')$ of infinite order and apply Definition 3.1.2(ii) with this t. Then by Definition 3.1.2(ii) we have $D'' = C(Z_M(D', t_1))$ for $t_1 \in tD'$. Since $D'' \in \tilde{\Pi}_m(M, T)$ and since by the inductive assumptions $\tilde{\pi}_m$ and $\tilde{\pi}'_m$ are inverse, it follows that $T'' = C(Z_H(D', t_1)^0 = C(Z_H(T'''))^0$, i.e., Definition 3.1.1(ii) holds. This shows that $\tilde{\pi}_{m+1}$ is well defined and concludes the proof of (i).

Let us prove (ii). Take $T' \in \tilde{\Pi}_{m+1}(H, T)$. Then by a property of big subgroups $T' = C(Z_H(C(Z_M(M \cap T'))))^0$. So $\tilde{\pi}_{m+1} \circ \tilde{\pi}'_{m+1} = 1$. Take now $D' \in \tilde{\Pi}_{m+1}(M, T)$. Let $R = C(Z_H(D'))$. Then $\tilde{\pi}_{m+1}(D') = R^0$. Also set $R_1 = C(Z_H(R^0))$. Both groups R/R^0 and R_1/R^0 are finite. Thus R/R_1 is finite. Set $D_1 = M \cap R_1$. Then since $R_1 \supset R^0$, and $M \cap R^0$ is dense in R^0, we have $Z_M(D_1) = Z_M(R^0)$ whence $D_1 = C(Z_M(D_1))$. From R/R_1 finite, it follows that D'/D_1 is periodic. Take D'' as in Definition 3.1.2(ii) for D'. Then $D'' \supset D_1$.

Take $t \in D''/D_1$ of infinite order. Then the image of t in D''/D' is also of infinite order (since D'/D_1 is periodic). Take $t_1 \in tD_1$ and consider $T''' = C(Z_H(D_1, t_1))^0$. We have $T''' \neq T'$, T''' is deined over k. So by Definition 3.1.1(ii) $C(Z_H(T'''))^0 = T''$ which implies that $C(Z_M(D_1, D_1))^0 = T''$ which implies that $C(Z_M(D_1, t_1)) = D''$. So D_1 satisfies the assumptions of Definition 3.1.2(ii). Therefore by Definition 3.1.2(iii) we must have $D_1 = D'$. This proves that $\tilde{\pi}'_{m+1} \circ \tilde{\pi}_{m+1} = 1$.

3.1.5. Let us define a mapping $\pi_i: \tilde{\Pi}_i(H, T) \to \Pi_i(H, T)$ by $\pi_i(T') = \Sigma_T(T')$ and a mapping $\pi'_i: \Pi_i(H, T) \to \tilde{\Pi}_i(H, T)$ by $\pi_i'(\Sigma') = T(\Sigma')^0$.

PROPOSITION. *The above mappings*

(i) *are well defined,*

(ii) *are inverses to one another.*

In particular, they establish an identification of $\tilde{\Pi}_i(H, T)$ with $\Pi_i(H, T)$.

Proof. The above statements are true if $i = 1$. Let us assume that they are true $i \leqslant m$ and let us prove by induction that they are true also for $i = m + 1$.

If $T' \in \tilde{\Pi}_{m+1}(H, T)$ and $\Sigma' = \mathbf{Q}\Sigma_T(T') \cap \Sigma_T$ it follows from the fact that T' is connected and does not contain subgroups of finite index that $\Sigma' = \Sigma_T(T')$, i.e., Σ' is saturated. It is Γ-stable since T' is defined over k. Take now T'' as in Definition 3.1.1(ii) and set $\Sigma'' = \Sigma_T(T'')$. By the induction assumption $\Sigma'' \in \Pi_m(H, T)$. Now take Σ''' which is saturated, Γ-stable and such that $\Sigma' \supseteq \Sigma''' \supset \Sigma''$, $\Sigma'' \neq \Sigma'''$. Then $T''' = T(\Sigma''')^0$ is a k-subtorus of T, $T''' \neq T'$, and $T''' = C(Z_H(T'''))^0 \neq T''$, a contradiction. This shows that Definition 3.1.3 holds for Σ', i.e., $\Sigma' \in \Pi_{m+1}(H, T)$, i.e., π_{m+1} is well defined.

If $\Sigma' \in \Pi_{m+1}(H, T)$, then Σ' is saturated and therefore $T(\Sigma')^0 = C(Z_H(T(\Sigma')))^0$. Set $T' = T(\Sigma')^0$. Take Σ'' as in Definition 3.1.3 and set $T'' = T(\Sigma'')^0$. Then, by the inductive assumptions, $T'' \in \tilde{\Pi}_m(H, T)$. Take a k-subtorus T''' of T'' such that $T'' \supseteq T''' \supseteq T'$, $T''' \neq T'$. Then $\Sigma_T(T''')$ coincides with Σ'' (by Definition 3.1.3) when $C(Z_H(T'''))^0 = T''$. So the conditions of Definition 3.1.1 hold for T', i.e., $T' \in \tilde{\Pi}_{m+1}(H, T)$, i.e., π'_{m+1} is well defined. This proves that part (i) of our lemma holds.

The fact that π_{m+1} and π'_{m+1} are inverse to one another is trivial (and was actually established above).

3.1.6. LEMMA. *Every saturated Γ-stable subset of Σ_T belongs to one of $\Pi_i(H, T)$.*

Proof. Let $\tilde{\Sigma}$ be such a subset. If $\tilde{\Sigma} = \phi$, then $\tilde{\Sigma} \in \Pi_1(H, T)$. Suppose that $\tilde{\Sigma} \neq \phi$. Then $\tilde{\Sigma}$ contains some maximal saturated Γ-stable subset Σ_1. It can be inductively assumed that $\Sigma_1 \in \Pi_m(H, T)$. Then $\tilde{\Sigma}$ satisfies conditions of Definition 3.1.3 whence $\tilde{\Sigma} \in \Pi_{m+1}(H, T)$.

3.2. Let k be an infinite field, let H be a reductive k-group, and let M be a big subgroup of $H(k)$. Let k' be another infinite field, let H' be a reductive k'-group, and let M' be a big subgroup of $H'(k')$. Let $\alpha: M \to M'$ be an isomorphism.

3.2.1. PROPOSITION. *Let T be a maximal k-subtorus of H and $T' = Z_{H'}(\alpha(T \cap M))$ (i.e., $T' = \alpha_*(T)$ in the notation of Corollary 2.4.5). Then α establishes bijections:*

$$\alpha_*: \tilde{\Pi}_i(H, T) \to \tilde{\Pi}_i(H'', T'),$$

$$\alpha^*: \Pi_i(H', T') \to \Pi_i(H, T).$$

Proof. It is immediate from Corollary 2.4.5 and the definitions of $\tilde{\Pi}_i(M, T)$ and $\tilde{\Pi}_i(M', T')$ that α establishes a bijection $\tilde{\Pi}_i(M, T) \to \tilde{\Pi}_i(M', T')$. Using identifications of Propositions 3.1.4 and 3.1.5 we recover α^* ınd α_*.

4. COINCIDENCE OF TYPES

Let k be an infibite field, let K be its separable quadratic extension, and let G be an absolutely almost simple anisotropic k-group split over K (i.e., K-admissible). Let k' be another infinite field and let G' be an absolutely almost simple k'-group. Let M and M' be big subgroups of $G(k)$ and $G'(k')$, respectively. Finally, let $\alpha: M \to M'$ be a group isomorphism. Our aim in this section is to prove the following theorem.

4.1. THEOREM. (i) *If $rkG \geqslant 2$, then there exists a separable quadratic extension K' of k' such that G' is split by K'. Moreover K' splits every torus $\alpha_*(T)$ where T is a K-admissible torus of G.*

(ii) *If G is not of type G_2, then there exists a (unique up to a sign) isomorphism of root systems $\tilde{\alpha}^*: \Sigma' \to \Sigma$ such that*

$$\alpha^*(\{\pm a'\}) = \{\pm \tilde{\alpha}^*(a')\} \qquad \textit{for } a' \in \Sigma'.$$

(iii) *If G is of type G_2, then G' is also of type G_2.*

4.2. The proof of Theorem 4.1 is quite long and cumbersome. Parts (i) and (iii) are proved in Section 4.5 and part (ii) in Section 4.6. We need the following notation. We pick a K-admissible torus T in G and we set $\Sigma = \Sigma(G, T)$, $\Gamma = \{1, \sigma\} = \Gamma(T)$, $\tilde{\Pi}_i = \tilde{\Pi}_i(G, T)$, $\Pi_i = \Pi_i(G, T)$, $T' = \alpha_*(T)$, $\Gamma' = \Gamma(T')$, $\tilde{\Pi}'_i = \tilde{\Pi}_i(G', T')$, and $\Pi'_i = \Pi_i(G', T')$.

By Lemma 3.1.6 and Proposition 1.1(ii) the set Π_i consists of all saturated subsystems of rank $i-1$ in Σ. In particular, Π_2 is the set of pairs $\{\pm a\}$,

$a \in \Sigma$. For $a \in \Sigma$ we set $\Sigma'(a) = \alpha^{*-1}(\{\pm a\})$ (cf. Proposition 3.2.1). Then $\Sigma'(a) \in \Pi'_2$.

For a subset $\tilde{\Sigma}$ such that the set of roots of $G(\tilde{\Sigma})$ with respect to T is $\tilde{\Sigma}$ (even if it is *not* a root subsystem) we identify $\tilde{\Sigma} \subset \Sigma$ with $\Sigma(G(\tilde{\Sigma}), T)$. We denote by $\tilde{\Sigma}'$ the set of roots of the Zariski closure of $\alpha(G(\tilde{\Sigma}) \cap M)$ with respect to T' (which makes sense since $G(\tilde{\Sigma})$ is normalized by T and therefore the Zariski closure of $\alpha(G(\tilde{\Sigma}) \cap M)$ is normalized by $T' = \alpha_*(T)$). Let $\tilde{w}_0$ be the largest element of the Weyl group of $\tilde{\Sigma}$, let $\tilde{w}_0 = s_{a_1} \cdot s_{a_2} \cdots s_{a_d}$, where a_i are simple roots of $\tilde{\Sigma}$, $d = |\tilde{\Sigma}^+|$, and let $\tilde{n}_i$ be the maximum of $|\Sigma'(a_i)|$ when T varies over the set of K-admissible tori of $G(\tilde{\Sigma})$. Let $\tilde{N} = |\Sigma'(a_1)| + \sum_{j=2}^{d} \tilde{n}_j$.

4.2.1. PROPOSITION. *The set of k'-tori $\alpha_*(\tilde{T})$, where $\tilde{T}$ varies over the set of K-admissible tori of $G(\tilde{\Sigma})$, is contained in a closed subset V of G' with* $\dim \tilde{V} \leqslant \dim T' + \tilde{N}$.

We shall prove this proposition in Section 4.2.4.

4.2.2. Let $\tilde{n}$ be the maximum of $|\Sigma'(a)|$ when a ranges over $\tilde{\Sigma}$ and T ranges over the set of K-admissible tori of $G(\tilde{\Sigma})$. We denote by n the number $\tilde{n}$ computed for $\tilde{\Sigma} = \Sigma$. Let $\bar{\tilde{n}} = \min_{i=1,\dots,d} |\Sigma'(a_i)|$ and let $\bar{n}$ be the corresponding number for Σ.

COROLLARY. $\dim \tilde{V} \leqslant \dim T' + \bar{\tilde{n}} + (|\tilde{\Sigma}^+| - 1) \cdot \tilde{n}$.

Proof. We have $\tilde{N} \leqslant |\Sigma'(a_1)| + (|\tilde{\Sigma}^+| - 1)\,\tilde{n}$. Since the reduced expression for w_0 can be chosen to begin with any simple root, we can replace $|\Sigma'(a_1)|$ by $\bar{\tilde{n}}$ in the preceding formula. This proves the corollary.

4.2.3. COROLLARIES. (i) *If $\bar{\tilde{n}} + (|\tilde{\Sigma}^+| - 1) \cdot \tilde{n} < |\Sigma'|$, then $\alpha(M \cap G(\tilde{\Sigma}))$ is not dense in G'*;

(ii) $\bar{n} + (|\Sigma^+| - 1) \cdot n \geqslant |\Sigma'|$;

(iii) $\bar{\tilde{n}} + (|\tilde{\Sigma}^+| - 1) \cdot \tilde{n} \geqslant |\tilde{\Sigma}'|$.

Proof. Assume that $\bar{\tilde{n}} + (|\tilde{\Sigma}^+| - 1) \cdot \tilde{n} < |\Sigma'|$. Then $\dim \tilde{V} < \dim G'$. Suppose that $\alpha(M \cap G(\tilde{\Sigma}))$ is dense in G'. Since the set of K-admissible tori of $G(\tilde{\Sigma})$ is stable under conjugation by $M \cap G(\tilde{\Sigma})$, it follows that $\tilde{V}$ contains a subset which is stable under conjugation by $\alpha(M \cap G(\tilde{\Sigma}))$ and which contains a maximal torus T'. Since $\alpha(M \cap G(\tilde{\Sigma}))$ is dense in G', it follows that $\tilde{V}$ contains the set $\bigcup_{g \in G'} gT'g^{-1}$ whence $\tilde{V} = G'$, a contradiction with $\dim \tilde{V} < \dim G'$. To prove (ii) assume that $\bar{n} + (|\Sigma^+| - 1)\, n < |\Sigma'|$ and apply (i) with $\tilde{\Sigma} = \Sigma$, $G(\tilde{\Sigma}) = G$. Then we must conclude that $M' = \alpha(M)$ is not dense in G', a contradiction.

To prove (iii) assume that $\bar{\tilde{n}} + (|\tilde{\Sigma} + | - 1)\,\tilde{n} < |\tilde{\Sigma}'|$. Let $H = G(\tilde{\Sigma}) \cdot T$ and let H' be the connected component of the Zariski closure of $\alpha(H \cap M)$. Thus

H' is the Zariski closure of a subgroup $N \subset G(\tilde{\Sigma}) \cap M$ of finite index. By Corollary 2.4.7 the group H' is reductive. Since $\alpha(N)$ is dense in H', the set $\bigcup_{h \in \alpha(N)} hT'h^{-1}$ is dense in H'. But these sets are contained in $\tilde{V}$ and by 4.2.2 we have $\dim \tilde{V} \leqslant \tilde{\tilde{n}} + (|\tilde{\Sigma}^+| - 1)\tilde{n} + \dim T' < \dim H'$. Thus we have a contradiction, which concludes the proof of our corollaries.

4.2.4. Let $\Sigma'_{i1} = \Sigma'(a_i), \Sigma'_{i2}, \ldots, \Sigma'_{i,q(i)}$ be the set of all subsystems of Σ' of cardinality $\leqslant \tilde{n}_i$. Let S be the set of ordered sequences $s = (j_1, j_2, \ldots, j_d)$ such that $j_1 = 1$, $1 \leqslant j_i \leqslant q(i)$. For $s \in S$ we set $\tilde{R}(s) = G'(\Sigma'_{11})\, G'(\Sigma'_{2j_2}) \cdots G'(\Sigma'_{dj_d})$ and $\tilde{V}(s) = \{gTg^{-1}, g \in \tilde{R}(s)\}$.

PROPOSITION. (i) *For any K-admissible subtorus $\tilde{T}$ of $G(\tilde{\Sigma})$ there exists $s \in S$ and $h \in \tilde{R}(s)$ such that $\alpha_*(\tilde{T}) = h(\alpha_*(T))\, h^{-1}$*,

(ii) $\dim \tilde{R}(s) \leqslant |\Sigma'(a_1)| + \Sigma_{i=2}^{d} \tilde{n}_i + \dim T'$,

(iii) $\dim \tilde{V}(s) \leqslant |\Sigma'(a_1)| + \Sigma_{i=2}^{d} \tilde{n}_i + \dim T'$.

We have immediately

COROLLARY. *Proposition* 4.2.1 *holds with*

$$\tilde{V} = \bigcup_{s \in S} \tilde{V}(s).$$

Thus it remains to prove Proposition 4.2.4.

Proof of Proposition. Statement (ii) is the direct consequence of the assumption $|\Sigma'_{ij}| \leqslant \tilde{n}_i$ and $j_1 = 1$. Statement (iii) immediately follows from (ii).

Let us prove (i). Let $T = T_0, T_1, \ldots, T_d = \tilde{T}$ a sequence of associated K-admissible tori of $G(\tilde{\Sigma})$ constructed in 1.4.2 which connects T and $\tilde{T}$. Let $\tilde{\Sigma} = \Sigma(G(\tilde{\Sigma}), T_i)$, $T'_i = \alpha_*(T_i)$, and $\Sigma'^i = \Sigma(G', T'_i)$. For $a \in \tilde{\Sigma}^i$ denote by $\Sigma'^i(a)$ the set $\alpha^{*-1}(\{\pm a\})$, where α^*: $\Pi_2(G', T'_i) \to \Pi_2(G, T_i)$ (cf. Proposition 3.2.1). Let $G^i(a)$, $a \in \tilde{\Sigma}^i$, denote the group constructed with respect to T_i and by $G'^i(\Sigma'^i(a))$ the analogous subgroup in G'.

We have $T_{i+1} \subset T_i \cdot G^i(a_i)$ for $a_i \in \tilde{\Sigma}^i$. Therefore $T'_{i+1} \subset T'_i \cdot G'^i(\Sigma^{i'}(a_i))$. Therefore there exists $h_i \in G'^i(\Sigma^{i'}(a_i))$ such that $T'_{i+1} = h_i T'_i h_i^{-1}$. Hence $T'_{i+1} = (h_i \cdots h_0)\, T'_0 (h_i \cdots h_0)^{-1}$.

Therefore conjugation by $\tilde{h}_i = h_i \cdots h_0$ induces a bijection of the set of subgroups of G' normalized by T'_0 onto the set of subgroups normalized by T'_i. In particular, there exists a subsystem $\Sigma'(i)$ of Σ such that $G'^i(\Sigma^{i'}(a_i)) = \tilde{h}_{i-1} G'(\Sigma'(i))\, \tilde{h}_{i-1}^{-1}$. Clearly $|\Sigma'(i)| = |\Sigma^{i'}(a_i)|$ and by our choice of $\tilde{n}$ we have $|\Sigma^{i'}(a_i)| \leqslant \tilde{n}$.

Now our aim is to show that $\tilde{h}_i \in G'(\Sigma'(0)) \cdots G'(\Sigma'(i))$. Since $\Sigma'(0) = \Sigma'(a_1)$ and since every $\Sigma'(k)$ coincides with some Σ'_{i,j_i}, assertion (i) will then follow.

Since $G^{i'}(\Sigma^{i'}(a_i)) = \bar{h}_i G'(\Sigma'(i)) \bar{h}_i^{-1}$, we can take $h_i' \in G'(\Sigma'(i))$ so that $h_0' = h_0$, $h_i = \bar{h}_{i-1} h_i' \bar{h}_{i-1}^{-1}$. Let us prove by induction that $\bar{h}_i = h_0' \cdots h_i'$. For $i = 0$ this is true. Suppose it is true for i, that is, $\bar{h}_i = h_0' \cdots h_i'$. Let us prove this for $\bar{h}_{i+1}$. We have

$$\bar{h}_{i+1} = h_{i+1} \cdot \bar{h}_i = \bar{h}_i h_{i+1}' \bar{h}_i^{-1} \cdot \bar{h}_i = \bar{h}_i \cdot h_{i+1}'$$

whence our assertion.

4.3. Now we are going to study the Γ'-module $\mathbf{Q}\Sigma'$. We use notation of the beginning of Section 4.2.

4.3.1. PROPOSITION. (i) *If* $a_1, \dots, a_m \in \Sigma$ *is a* $\mathbf{Q}$*-basis of* $\mathbf{Q}\Sigma$, *then* $\mathbf{Q}\Sigma' = \bigoplus_{i=1}^m \mathbf{Q}\Sigma'(a_i)$ *(a direct sum of* Γ'*-modules)*;

(ii) Γ'*-modules* $\mathbf{Q}\Sigma'(a)$ *and* $\mathbf{Q}\Sigma'(b)$ *are isomorphic for any* $a, b \in \Sigma$;

(iii) *if* T_1, T_2 *are K-admissible tori of G, then the minimal splitting fields of* $\alpha_*(T_1)$ *and* $\alpha_*(T_2)$ *are the same.*

4.3.2. *Proof of* (i). Take $D_i \in \tilde{\Pi}_i(M, T)$ such that the roots of $Z_G(D_i)$ are $\{\pm a_i\}$. Set $M_i = \mathscr{D}^2(Z_M(D_i))$, $\tilde{M} = \langle M_i,\ i = 1, \dots, m\rangle$. Then $Z_G(\tilde{M})$ is finite. Therefore $Z_{M'}(\alpha(\tilde{M}))$ is finite. Let H_i' be the Zariski closure of $\alpha(M_i)$ in G'; since $M_i = \mathscr{D}^2(Z_M(D_i))$, it follows from Lemma 2.1(v), Proposition 2.4.4, and [14, 4.4] that H_i' is connected, reductive, and defined over k'. All H_i' are normalized by $T' = \alpha_*(T)$ and $\Sigma'(a_i) = \Sigma(H_i', T')$.

Let $\tilde{\Sigma}' = \bigcup_i \Sigma'(a_i)$, $\tilde{T}' = \{t \in T \mid \alpha(t) = 1$ for all $\alpha \in \tilde{\Sigma}'\}$. The group $\tilde{T}'^0$ must be trivial since otherwise it would be a k-torus and then the group $M' \cap \tilde{T}'^0 (\subset Z_{M'}(\alpha(\tilde{M})))$ would be infinite by the property of big groups in contradiction with finiteness of $Z_{M'}(\alpha(\tilde{M}))$. But since $\tilde{T}'$ is finite, it follows that $\tilde{\Sigma}$ spans $\mathbf{Q}\Sigma'$ over $\mathbf{Q}$. Thus $\mathbf{Q}\Sigma' = \sum_{i \leqslant m} \mathbf{Q}\Sigma'(a_i)$.

Suppose that this sum is not direct. Set $\Sigma_i' = \Sigma'(a_i)$, $\Sigma_i = \{\pm a_i\}$. Then for some q we have $\mathbf{Q}\Sigma_q' \cap \sum_{i \neq q} \mathbf{Q}\Sigma_i' \neq 0$. Set $\tilde{\Sigma} = (\sum_{i \neq q} \mathbf{Q}\Sigma_i) \cap \Sigma$, $\tilde{\Sigma}' = (\sum_{i \neq q} \mathbf{Q}\Sigma_i') \cap \Sigma'$, $\tilde{T} = T(\tilde{\Sigma})$, $T_q = T(\{\pm a_q\})$, $T_q' = T'(\Sigma_q')$, and $\tilde{T}' = T'(\tilde{\Sigma}')$. Then we have $T_q \cdot \tilde{T} = T$. We have therefore by the property of big groups $(M \cap T)/(T_q \cap M)(\tilde{T} \cap M)$ is periodic but $(M' \cap T')/(T_q' \cap M')(\tilde{T}' \cap M')$ is not periodic. Since $M' \cap T' = \alpha(M \cap T)$, $T_q' \cap M' = \alpha(T_q \cap M)$, and $\tilde{T}' \cap M' = \alpha(\tilde{T} \cap M)$, we have a contradiction. We arrived at this contradiction on the basis of the assumption that the sum is not direct. Hence it is direct.

4.3.3. *Proof of* (ii). Let $\{a, a_2, \dots, a_m\}$ and $\{b, a_2, \dots, a_m\}$ be two sets of roots, such that both of them are $\mathbf{Q}$-bases of $\mathbf{Q}\Sigma$. Then

$$\mathbf{Q}\Sigma' = \mathbf{Q}\Sigma'(a) \oplus \bigoplus_{i \geqslant 2} \mathbf{Q}\Sigma'(a_i) \cong \mathbf{Q}\Sigma'(b) \oplus \bigoplus_{i \geqslant 2} \mathbf{Q}\Sigma'(a_i)$$

(isomorphisms of Γ'-modules). It follows that

$$\mathbf{Q}\Sigma'(a) \simeq \mathbf{Q}\Sigma'(b).$$

Let now $\Delta = \{b_1, ..., b_m\}$ be a system of simple roots in Σ. Set $\Delta_i = \Delta - b_i$, $\Sigma_i = \mathbf{Q}\Delta_i \cap \Sigma$. Let b be the maximal root in Σ (with respect to Δ). Then $b \notin \Sigma_i$ for all i. Let a be an arbitrary root from $\Sigma - \Sigma_i$. Then $a, b_1, ..., \hat{b}_i, ..., b_m$ is a basis of $\mathbf{Q}\Sigma$. By the preceding assertion we have therefore $\Sigma'(a) \simeq \Sigma'(b)$ for all $a \in \Sigma - \Sigma_i$. Since i can be taken arbitrary, $\Sigma'(a) \simeq \Sigma'(b)$ for all $a \in \Sigma$, whence our assertion.

4.3.4. *Proof of* (iii). By Section 1.4 or [19, no. 7] it is sufficient to prove (iii) in the case when $T_1 = T$, $T_2 \subset G(a)T$ for some $a \in \Sigma$. Set $T'_i = \alpha_*(T_i)$. Then $T'_1 \cap T'_2 = T'(\Sigma'(a))$. Since the representation of $\Gamma(T'_i)$ is a direct sum of equivalent representations and since $\Gamma(T'_1)$ and $\Gamma(T'_2)$ act as $\Gamma(T'_1 \cap T'_2)$ on $\mathbf{Q} \cdot X(T'_1 \cap T'_2)$, we have $\Gamma(T'_i) = \Gamma(T'_1 \cap T'_2)$, as asserted.

4.4. Now we will consider the case when Σ is of rank 2, i.e., when Σ is of type A_2, B_2, G_2. The argument below consists of case considerations and is based on Proposition 4.2.1 and its corollaries.

4.4.1. PROPOSITION. *If* $\operatorname{rk} G = 2$, *then* $\operatorname{rk} G' = 2$ *and* Σ *and* Σ' *are of the same type.*

4.4.2. *Proof.* By Proposition 4.3.1(ii) we known that $\dim \mathbf{Q}\Sigma'(a)$ does not depend on $a \in \Sigma$. Set $d = \dim \mathbf{Q}\Sigma'(a)$. Then Proposition 4.3.1(i) says that $\operatorname{rk} \Sigma' = 2d$. Now Corollary 4.2.3 says that $\bar{n} + (|\Sigma^+| - 1) \cdot n \geqslant |\Sigma'|$. In particular, we have a rougher inequality $|\Sigma^+| \cdot n \geqslant |\Sigma'|$, where n is the maximum cardinality of the subsystems of rank d in Σ'. Let $\bar{\Sigma}'$ be a subsystem of rank d and maximum cardinality. Let us tabulate the relevant information.

Type Σ'	A_{2d}	B_{2d}	C_{2d}	D_4	$D_{2d}, d > 2$	G_2	F_4	E_6	E_8
Type $\bar{\Sigma}'$	A_d	B_d	C_d	A_2	D_d	A_1	B_2	A_3	D_4
$\lvert\Sigma'\rvert$	$2d(2d+1)$	$8d^2$	$8d^2$	24	$8d^2 - 4d$	12	48	72	240
$\lvert\bar{\Sigma}'\rvert$	$d(d+1)$	$2d^2$	$2d^2$	6	$2d^2 - 2d$	2	8	12	24
Σ of type A_2, $\lvert\Sigma^+\rvert \cdot \lvert\bar{\Sigma}'\rvert$	$3d(d+1)$	$6d^2$	$6d^2$	18	$6(d^2 - d)$	6	24	36	72
Σ of type B_2, $\lvert\Sigma^+\rvert \cdot \lvert\bar{\Sigma}'\rvert$	$4d(d+1)$	$8d^2$	$8d^2$	24	$8(d^2 - d)$	8	32	48	96
Σ of type G_2, $\lvert\Sigma^+\rvert \cdot \lvert\bar{\Sigma}'\rvert$	$6d(d+1)$	$12d^2$	$12d^2$	36	$12(d^2 - d)$	12	48	72	144

4.4.2. *Case* Σ of type A_2. It is seen from the table that the inequality $|\Sigma^+| \cdot n \geqslant |\Sigma'|$ is possible only if Σ' is of type A_{2d} with $d=1$. This proves Proposition 4.4.1 for Σ of type A_2.

4.4.3. *Case* Σ of type B_2, Suppose that Σ' is not of type B_2. Then we see from the table that for Σ' we have the following possibilities: A_{2d}, $B_{2d}(d \geqslant 1)$, $C_{2d}(d \geqslant 1)$, and D_4. Let $\tilde{a}$ be a long root of Σ and let $\tilde{b}$ be a (long) root orthogonal to $\tilde{a}$. Then $G(\tilde{a})$ and $G(\tilde{b})$ commute whence $G'(\Sigma'(\tilde{a}))$ and $G'(\Sigma'(\tilde{b}))$ commute. They are semi-simple by Corollary 2.4.7. So G' contains two semi-simple subgroups of rank d which commute.

If Σ' is of type A_{2d}, this is impossible.

If Σ' is of type D_4, then $G'(\Sigma'(\tilde{a}))$ and $G'(\Sigma'(\tilde{b}))$ are of type $A_1 \times A_1$ whence $\bar{n}=4$. Now use Corollary 4.2.3(ii). We have $\bar{n}+(|\Sigma^+|-1) \cdot n \leqslant 4+3 \cdot 6=22<|\Sigma'|=24$, a contradiction.

If Σ' is of type B_{2d}, $d \geqslant 1$, then B_d is the only subsystem of rank d and cardinality $2d^2$. We conclude from Corollary 4.2.3(ii) that $\Sigma'(a)$ is of type B_d for any $a \in \Sigma$. Assume that $d>1$. The number of short roots in B_d is d and the number of short roots in B_{2d} is $2d$. Thus B_{2d} cannot be covered by disjoint subsustems of type B_d (and they must be disjoint by Proposition 4.3.1(i)) unless $d=1$.

If Σ' is of type C_{2d}, $d>1$, then the same argument as above as above (for B_{2d}) applies. Namely, each $\Sigma'(a)$ must be of type C_d. The number of long roots in C_d is d and in C_{2d} it is $2d$ whence C_{2d} cannot be covered by disjoint subsystems of type C_d.

This proves Proposition 4.4.1 for Σ of type B_2.

4.4.4. *Case* Σ of type G_2. Let $\tilde{\Sigma}$ be the subsystem of long roots in Σ. Then $G(\tilde{\Sigma})$ is of type A_2. Since $3n=|\tilde{\Sigma}^+| \cdot n<|\Sigma'|$, it follows from Corollary 4.2.3(i) that $\alpha(M \cap G(\tilde{\Sigma}))$ is not dense in G'. Let H' be the connected component of its closure. It is a semi-simple subgroup of maximal rank (by Corollary 2.4.7). By Corollary 2.4.9, $H'=R_{k''/k'}H''$ where H'' is absolutely almost simple. Applying Corollary 4.2.3(iii) to α: $M \cap G(\tilde{\Sigma}) \to H''$ and using the line of our table corresponding to A_2 we derive that H'' is of type A_2. Thus $\tilde{\Sigma}'$ is of type $A_2 \times A_2 \times \cdots \times A_2$ (d times). This is possible only if Σ' is of type G_2, F_4, E_6, E_8.

The case G_2 does not contradict our statement.

The case F_4 is rejected as follows. In a group of type F_4 two components in a subsystem of type $A_2 \times A_2$ consist of roots of different length. Thus $G'(\tilde{\Sigma}')$ is not almost k'-simple. This is a contradiction with Lemma 2.4.8.

If Σ' is of type E_6, then we observe that $\Sigma'(a)$ is of type $A_1 \times A_1 \times A_1$ for any $a \in \tilde{\Sigma}$. Therefore $\bar{n} \leqslant 6$. Thus $\bar{n}+(|\Sigma^+|-1)n=6+5 \cdot 12=66<72$. Now Corollary 4.2.3(ii) gives the desired contradiction.

The case Σ' of type E_8 is rejected by the table.

This completes the proof of Proposition 4.4.1.

4.5. **Proposition.** *If* rk $G \geqslant 2$, *then there exists a separable quadratic extension* K' *of* k' *such that* G' *is split by* K'. *Moreover*,

(i) rk $G' =$ rk G,

(ii) $|\Sigma| = |\Sigma'|$,

(iii) K' *splits the torus* $\alpha_*(\tilde{T})$ *if* $\tilde{T}$ *is a* K-*admissible torus of* G.

4.5.1. Let $\tilde{\Sigma} \subset \Sigma$ be a connected saturated subsystem of rank 2. Then $\tilde{\Sigma} \in \Pi_3$ (by Lemma 3.1.6 and Proposition 1.1(ii)). Let $\tilde{T} \in \tilde{\Pi}_3$ be the corresponding k-torus (cf. 3.1.5). Set $H = Z_G(\tilde{T})/C(Z_G(\tilde{T}))$, $H' = Z_{G'}(\alpha_*(\tilde{T}))/C(Z_{G'}(\alpha_*(\tilde{T}))))$. Now Corollary 2.4.9 is applicable to the isomorphism α: $(Z_G(\tilde{T}) \cap M)/C(Z_G(\tilde{T}) \cap M) \to (Z_{G'}(\alpha_*(\tilde{T}) \cap M')/C(Z_{G'}(\alpha_*(\tilde{T})) \cap M)$. Next we apply 4.4 and deduce that $\tilde{\Sigma}' = \alpha^{*-1}(\tilde{\Sigma})$ consists of d copies of $\tilde{\Sigma}$.

4.5.2. If Σ is of type $B_n(n \geqslant 2)$, $C_n(n \geqslant 2)$, F_4 we can apply the above considerations to a sybsystem $\tilde{\Sigma}$ of type B_2 in Σ. Then it will follow that Σ' contains a saturated subsystem of type $B_2 \times \cdots \times B_2$ (d times). This is possible only if $d = 1$ (since no connected system of simple roots contains two subsystems of type B_2).

4.5.3. The conclusions of 4.5.1 imply that $\Sigma'(a)$ is of type $A_1 \times \cdots \times A_1$ (d times). By Corollary 4.2.3(ii) we must have $|\Sigma| \cdot d \geqslant |\Sigma'|$. Here Σ is of rank n and Σ' is of rank nd.

This clearly is impossible if Σ is of type A_n.

If Σ is of type D_n such a relation may be possible only if Σ' is of type A_{nd}. But then $|\Sigma'| = nd(nd+1) \leqslant d \cdot 2(n^2 - n) = d \cdot |\Sigma|$ reduces to $n^2d^2 \leqslant 2n^2d - 3nd$, i.e., to $n^2d^2 < 2n^2d$, i.e., to $d < 2$, as required.

If Σ is of type E_n, $n = 6, 7, 8$, take in Σ a subsystem $\tilde{\Sigma}$ of type D_4. Then $\tilde{\Sigma} \in \Sigma_5$ (by Lemma 3.1.6 and Proposition 1.1(ii)). The same argument as in 4.5.1 shows that $\tilde{\Sigma}' = \alpha^{*-1}(\tilde{\Sigma})$ consists of d copies of saturated subsystems of type D_4. This is only possible if $d = 1$ (since otherwise a connected system of simple roots would contain two subsystems of type D_4).

4.5.4. We have shown that $d = 1$ for any choice of Σ. By Proposition 4.3.1(ii) it follows that Γ' is represented faithfully on a one-dimensional space $\mathbf{Q}a'$, $a' \in \Sigma'$. Now Propositions 1.2 and 2.4.1 yield Proposition 4.5(ii). Now our argument becomes reversible. We use Corollary 4.2.3(ii) to deduce that $|\Sigma| \geqslant |\Sigma'|$ and since Σ and Σ' play now a symmetric role, we have also $|\Sigma'| \geqslant |\Sigma|$, whence $|\Sigma| = |\Sigma'|$. This completes the proof of Proposition 4.5.

4.6. Proof of Theorem 4.1(ii), (iii).

4.6.1. Suppose that α^* preserves the lengths of roots (i.e., it maps long roots into long ones and short ones into short ones). Then it is immediate that the types of Σ and Σ' coincide. We can apply [23, Proposition 5.11.1] and obtain the result in this case. The above assumptions hold automatically if Σ has roots of only one length. To handle the cases B_n, C_n, F_4 we need more work.

4.6.2. Consider first the case B_2. Because of Proposition 4.5(i), (ii) we know that Σ' is also of type B_2. If char $k = 2$, we can apply [21, Theorem 0.3]. Indeed the big groups of the present paper are big also in the sense of [21] (but not necessary conversely). As remarked in [21, 0.1] the condition of that paper, namely, [21, 4.1], automatically holds for K-admissible groups. Now let us apply [21, Theorem 0.3]. Since G' is anisotropic (by Lemma 2.4.1) it follows that a special k'-isogeny β of [4, Theorem 0.3] must be an isomorphism (cf. [4, Proposition 3.1]). This proves our assertion for type B_2 if char $k = 2$. Since by Proposition 4.5 the roles of k and k' are now interchangeable, our assertion is proved also if char $k' = 2$.

Now assume that char $k \neq 2$ and char $k' \neq 2$. Let a, b be the positive long roots of Σ. Then $G(a)$ and $G(b)$ commute. Therefore $G'(\alpha^{*-1}(a))$ and $G(\alpha^{*-1}(b))$ commute. Since char $k' \neq 2$, this means that $\alpha^{*-1}(a)$ and $\alpha^{*-1}(b)$ are long roots of Σ'. Thus we are back in the conditions of 4.6.1.

4.6.3. Now let Σ be of type B_n, C_n, F_4. By 4.6.2 the restriction of α to any connected saturated subsystem of type B_2 in Σ preserves lengths. Therefore α^* preserves lengths and we are in conditions of 4.6.1. This concludes the proof of Theorem 4.1(ii).

4.6.4. If Σ is of type G_2, then Σ' is of type G_2 because of Proposition 4.5(i) and .5(ii). This concludes the proof of Theorem 4.1(iii).

5. Reduction to Rank 2

Let k and k' be infinite fields. Let H be a reductive k-group and let H' be a reductive k'-group. Let $PH = H/C(H)$, $PH' = H'/C(H')$. Let M be a big subgroup of H and let M' be a big subgroup of H'. Finally, let α: $M \to M'$ be a group isomorphism and let $P\alpha$: $PM \to PM'$ be the induced isomorphism.

5.1. Definitions. (i) We say that α is *good* if there exist a field isomorphism φ: $k \to k'$ and a k'-isomorphism of algebraic k'-groups β: ${}^{\varphi}(PH) \to PH'$ such that $P\alpha(h) = \beta(\varphi^0(h))$ for $h \in PM$.

(ii) We say that α is *very good* if there exist a field isomorphism φ: $k \to k'$, a k'-isomorphism of algebraic k'-groups β: ${}^{\varphi}H \to H'$, and a homomorphism γ: $M \to C(M')$ such that $\alpha(h) = \gamma(h) \cdot \beta(\varphi^0(h))$ for $h \in M$.

5.2. Let K be a separable quadratic extension of k and let G be a K-admissible group. Let G' be an absolutely amost simple k'-group. Let M be a big subgroup of G and let M' be a big subroup of G'. Let α: $M \to M'$ be a group isomorphism.

THEOREM. *Suppose* (i) rk $G > 2$,

(ii) *for every K-admissible torus T of G and every connected saturated subsystem* $\tilde{\Sigma}$ *of* $\Sigma(G, T)$ *of rank* 2 *the restriction*

$$\alpha\colon (G(\tilde{\Sigma}) \cdot T) \cap M \to (G'(\alpha^{*-1}(\tilde{\Sigma}))\, \alpha_*(T)) \cap M'$$

is good,

(iii) *G and G′ have isomorphic root data if G is not of type* D_{2n} *and G and G′ are simultaneously simply connected or adjoint if G is of type* D_{2n}.

Then α *is very good.*

The proof is given in several steps.

5.3. Let us take the following assumptions: k and k' are infinite fields, H a reductive anisotropic k-group, H' a reductive k'-group, M a big subgroup of H, M' a big subgroup of H', and $\alpha\colon M \to M'$ a group isomorphism.

5.3.1. LEMMA. (i) *Suppose that* α *is good. Then a field isomorphism* φ: $k \to k'$ *and a* k'*-isomorphism* β: ${}^{\varphi}(PH) \to PH'$ *such that* $(P\alpha)(h) = \beta(\varphi^0(h))$ *for* $h \in PM$ *are uniquely determined by* α.

(ii) *Suppose that* α *is very good and that H and H′ are semi-simple with isomorphic root data. Then a field isomorphism* $\tilde{\varphi}$: $k \to k'$, *a* k'*-isomorphism* $\tilde{\beta}$: ${}^{\varphi}H \to H'$, *and a homomorphism* $\tilde{\gamma}$: $M \to C(M')$ *such that* $\alpha(h) = \gamma(h) \cdot \beta(\varphi^0(h))$ *for* $h \in M$ *are uniquely determined by* α.

(iii) *In case* (ii) *we have, moreover,* $\tilde{\varphi} = \varphi$, $\beta = P\tilde{\beta}$.

Proof. Suppose that $\beta(\varphi^0(h)) = \beta_1(\varphi_1^0(h))$, $h \in PM$. Regrouping β, β_1, φ, φ_1 we can assume that there exist a k-automorphism β_2: $PH \to PH$ and a field automorphism φ_2: $k \to k$ such that $\beta_2(h) = \varphi_2^0(h)$ for $h \in PM$. Since PM is dense in PH and $\varphi_2(h) \in PH$ for $h \in PM$, it follows that PH is defined over the fixed field of φ_2, say k_0. The field k_0 cannot be finite since then H would be quasi-split over k_0. Let V be the variety of maximal tori of M. It is defined over k_0 [1, 7.7] and $V(k_0)$ is dense in V [1, 7.9]. By the property of big subgroups we have $\tilde{\beta}_2(v) = \varphi_2^0(v)$ for $v \in V(k)$ (where $\tilde{\beta}_2 \in (\mathrm{Aut}\, V)(k)$ is the automorphism induced by β_2).

In this situation (i) follows from the following result stated and proved by A. Borel.

LEMMA. *Let V be a k-variety,* k_0 *its smallest field of definition,* β: $V \to V$ *a regular automorphism of V, and* φ: $k \to k$ *a field automorphism. Suppose that* $V(k_0)$ *is Zariski-dense in V. If* $\beta(x) = \varphi^0(x)$ *for* $x \in V(k)$, *then* $\beta = 1$, $\varphi = 1$.

Proof. Since $\varphi^0(x) \in V$ for $x \in V(k)$ and $V(k)$ is dense, V is defined over fixed field of φ, whence φ acts trivially on $V(k_0)$. Therefore $\beta(x)=x$ for $x \in V(k_0)$ and since $V(k_0)$ is dense in V, $\beta=1$, as required.

Now let us prove (ii) and (iii). Let us go over to PH, PH', PM, PM' (note that since M and M' are dense in H and H', their centers are contained in the centers of H and H'). Then we have $\tilde{\varphi}=\varphi$ and $\beta=P\tilde{\beta}$. Since $P\tilde{\beta}=1$ implies $\tilde{\beta}=1$, we have from (i) unicity for $\tilde{\varphi}$ and $\tilde{\beta}$. The unicity for $\tilde{\gamma}$ follows.

5.3.4. LEMMA. *Suppose that α is good. Suppose that $\mathscr{D}H$ and $\mathscr{D}H'$ are absolutely almost simple. If $\mathscr{D}H$ and $\mathscr{D}H'$ have isomorphic root data and in addition are both simply connected or both adjoint if $\mathscr{D}H$ is of type D_{2n}, then α: $\mathscr{D}M \to \mathscr{D}M'$ is very good.*

Proof. Since α is good, there exist a field isomorphism φ: $k \to k'$ and a k'-isomorphism β: $PH \to PH'$ such that $P\alpha=\beta(\varphi^0(h))$ for $h \in PM$. From our assumptions on $\mathscr{D}H$, $\mathscr{D}H'$, it follows that there exists a k'-isomorphism $\tilde{\beta}$: ${}^{\varphi}(\mathscr{D}H) \to \mathscr{D}H'$ which induces β on $P(\mathscr{D}H)=PH$. (Indeed, β can certainly be lifted to a k'_s-isomorphism $\tilde{\beta}$: ${}^{\varphi}(\mathscr{D}H) \to \mathscr{D}H'$. But this $\tilde{\beta}$ cannot differ from its conjugates by $\mathrm{Gal}(\bar{k}'_s/k')$ since $P\tilde{\beta}={}^{\sigma}P\tilde{\beta}$ for $\sigma \in \mathrm{Gal}(k'_s/k')$.) Consider then $\tilde{\alpha}=\tilde{\beta} \circ \varphi^0$: $\mathscr{D}M \to \mathscr{D}M'$. We have $P\alpha=P\tilde{\alpha}$ on $\mathscr{D}M/C(\mathscr{D}M)$. Therefore

$$\tilde{\alpha}(h) \cdot \alpha(h^{-1})=\gamma(h) \in C(\mathscr{D}M'), \qquad h \in \mathscr{D}M.$$

Since $\tilde{\alpha}(h)=\alpha(h)\cdot\gamma(h)$, we have that γ: $\mathscr{D}M \to C(\mathscr{D}M')$ is a homomorphism. So α has the required form.

5.3.5. LEMMA. *Let H_1, H_2 be reductive k-subgroups of H such that $H_1 \cap H_2$ is a non-commutative reductive group containing a maximal k-torus of H. Let $M_i=H_i \cap M$ and let H'_i be the Zariski closure in H' of $M'_i=\alpha(M_i)$, $i=1,2$. Suppose that the isomorphisms α: $M_i \to M'_i$ are good for $i=1,2$ so that $(P\alpha)(h)=\beta_i(\varphi_i^0)(h))$ for $h \in PM_i$, $i=1,2$. Then $\varphi_1=\varphi_2$ and the restrictions of β_1 and β_2 to ${}^{\varphi}P(H_1 \cap H_2)$ coincide.*

Proof. It is immediate from Lemma 5.3.1(i) applied to $H_1 \cap H_2$. The application of Lemma 5.3.1(i) is justified by Lemma 2.3.3(ii).

5.4. The inductive procedure is based on a lemma about root systems.

Let Σ be a root system, Δ its subsystem of simple roots, and Σ^+ the corresponding system of positive roots. For any subsystem $\tilde{\Sigma}$ of Σ such that $\tilde{\Sigma}=-\tilde{\Sigma}$ we put $\tilde{\Sigma}^+=\Sigma^+ \cap \tilde{\Sigma}$ and we denote by $\Delta(\tilde{\Sigma})$ the system of simple roots of $\tilde{\Sigma}$ corresponding to $\tilde{\Sigma}^+$.

Let $\Delta_1, \Delta_2, \ldots, \Delta_m$ be connected sybsystems of rank 2 in Δ which satisfy

(i) $\Delta=\bigcup_{i \leqslant m} \Delta_i$,

(ii) for any pair $a, b \in \Delta$, there exist $i_1, \ldots, i_q \in [1, m]$ such that $a \in \Delta_{i_1}$, $b \in \Delta_{i_q}$, $|\Delta_{i_j} \cap \Delta_{i_{j+1}}|=1$.

Set $\Sigma_i=\mathbf{Q}\Delta_i \cap \Sigma$.

LEMMA. *There exist connected saturated subsystems* $\Sigma_{m+1},\ldots,\Sigma_n$ *of rank* 2 *in* Σ *such that*

(a) $\Delta(\Sigma_i)\subset\bigcup_{j<i}\Sigma_j$ *for* $i>m$,

(b) $\Sigma_i\not\subset\bigcup_{j<i}\Sigma_j$ *for* $i>m$,

(c) $\Sigma=\bigcup_{j\leqslant n}\Sigma_j$.

Proof. It can be assumed that $\tilde{\Delta}_q=\bigcup_{j\leqslant q}\Delta_j$ is a connected subsystem of Δ of rank d and that the rank of $\tilde{\Delta}_{q+1}=\Delta_{q+1}\cup\tilde{\Delta}_q$ is $d+1$. Set $\bar{\Sigma}_i=\mathbf{Q}\Delta_i\cap\Sigma$, $\tilde{\Sigma}_i=\mathbf{Q}\tilde{\Delta}_i\cap\Sigma$. Applying [23, Lemma 5.11.4] with $\Sigma=\tilde{\Sigma}_{q+1}$, $\tilde{\Sigma}==\tilde{\Sigma}_q$, $\Sigma_1=\bar{\Sigma}_{q+1}$ we can cover $\tilde{\Sigma}_{q+1}$ by subsystems, satisfying (a) and (b). When q runs through $[1,m-1]$ we get our assertion.

5.5. Let us take notation and assumptions of Theorem 5.2. These assumptions imply that for every K-admissible torus T of G and every connected saturated subsystem $\tilde{\Sigma}\subset\Sigma(G,T)$ of rank 2 the isomorphism α: $M\cap G(\tilde{\Sigma})\cdot T\to M'\cap G'(\alpha^{*-1}(\tilde{\Sigma}'))\cdot\alpha_*(T)$ is good. (The intersection $M\cap G(\tilde{\Sigma})\cdot T$ is defined abstractly as $Z_M(D)$ for some $D\in\tilde{\Pi}_3(M,T)$.) Let us denote this isomorphism by $\alpha(T,\tilde{\Sigma})$. By Lemma 5.3.4 there exist a field isomorphism $\varphi(T,\tilde{\Sigma})$: $k\to k'$, a k'-isomorphism of algebraic groups $\beta(T,\tilde{\Sigma})$: $G(\tilde{\Sigma})\to G'(\alpha_*^{-1}(\tilde{\Sigma}))$, and a homomorphism $\gamma(T,\tilde{\Sigma})$: $M\cap G(\tilde{\Sigma})\to C(M'\cap G'(\alpha^{*-1}(\tilde{\Sigma})))$ such that $[\alpha(T,\tilde{\Sigma})](h)=[\gamma(T,\tilde{\Sigma})](h)[(\beta(T,\tilde{\Sigma}))\circ\varphi^0(T,\tilde{\Sigma})(h)]$ for $h\in M\cap G(\tilde{\Sigma})\supseteq\mathscr{D}(Z_M(D))$.

5.5.1. LEMMA. (i) *$\varphi(T,\tilde{\Sigma})$ does not depend on $\tilde{\Sigma}$; its common value is denoted $\varphi(T)$.*

(ii) *There exists a k'-isomorphism $\beta(T)$: ${}^{\varphi(T)}G\to G'$ which coincides with $\beta(T,\tilde{\Sigma})$ on $G(\tilde{\Sigma})$ for all connected $\tilde{\Sigma}\in\Pi_3(G,T)$.*

Proof. Lemmas 5.4, 5.3.1, 5.3.4, and 5.3.5 show that $\varphi(T,\tilde{\Sigma})$ does not depend on $\tilde{\Sigma}$. Now let us replace k by $\varphi(T)(k)$ and G by ${}^{\varphi(T)}G$ (so that we shall not need to write ${}^{\varphi(T)}G$ in this proof). Again applying Lemmas 5.3.1, 5.3.4, and 5.3.5 we see that $\beta(T,\tilde{\Sigma})\mid G(a)$, $a\in\Delta$, does not depend on $\tilde{\Sigma}\ni a$. Let us denote this restriction by $\beta(T,a)$. Let Δ be the system of simple roots of Σ. Then the set $\alpha^{*-1}(\pm\Delta)$ contains a system $\Delta'=\tilde{\alpha}^{*-1}(\Delta)$ of simple roots of Σ' (by Theorem 4.1(ii)). Set $a'=\tilde{\alpha}^{*-1}(a)$. Since $\beta(T,a)(T\cap G(a))=\alpha_*(T)\cap G'(a')$, we have $\beta(T,a)\,x_a(t)=x'_{\varepsilon(a)a'}(t')$ where ε: $\Delta\to\{\pm1\}$. Replacing if necessary $\tilde{\alpha}$ by $-\tilde{\alpha}^*$ we can assume that $\varepsilon(a_0)=1$ for some $a_0\in\Delta$. Let us show that then $\varepsilon(a)=1$ for all $a\in\Delta$.

Let $\tilde{\Delta}$ be a connected subsystem of Δ of rank 2. Set $\tilde{\Sigma}=(\mathbf{Q}\tilde{\Delta})\cap\Sigma$. Since $\beta(T,a)$, $a\in\tilde{\Delta}$, is a restriction of $\beta(T,\tilde{\Sigma})$, it follows that $\beta(T,\tilde{\Sigma})^*$: $\tilde{\Sigma}'\to\tilde{\Sigma}$ differs from $\tilde{\alpha}^*$ at most by sign. Therefore $\varepsilon(a)=\varepsilon(b)$ for a, $b\in\tilde{\Delta}$. Since Δ can be covered by such subsystems $\tilde{\Delta}$ in such a way that every point is covered at least twice it follows that $\varepsilon(a) = \varepsilon(b)$ for any $a, b \in \Delta$. Since $\varepsilon(a_0) = 1$ we have $\varepsilon(a) = 1$ for all $a \in \Delta$, as desired.

Now we have a set of isomorphisms $\beta(T, a)$: $G(a) \to G'(a')$, $a \in \Delta$, such that $\beta^{*-1}(T, a)(a) = a'$. Applying 1.3 we get that there exists $\beta(T)$: $G \to G'$ which coincides with $\beta(T, a)$ on $G(a)$ for all $a \in \Delta$. Using Lemmas 5.4, 5.3.1, and 5.3.5 we establish successively that $\beta(T) \mid G(\Sigma_i) = \beta(T, \Sigma_i)$ (where Σ_i are from Lemma 5.4). We use, of course, the fact that $G(\Sigma_i)$ is generated by $G(a)$, $a \in \Delta(\Sigma_i)$, whence $\beta(T, \Sigma_i)$ is completely determined by its restriction to $G(a)$, $a \in \Delta(\Sigma_i)$. Thus $\beta(T) \mid G(a)$ must coincide with $\beta(\tilde{\Sigma}, T) \mid G(a)$ for all connected $\tilde{\Sigma} \in \tilde{\Pi}_3(G, T)$ such that $\tilde{\Sigma} \ni a$.

5.5.2. LEMMA. *$\varphi(T)$ and $\beta(T)$ do not depend on the choice of a K-admissible torus T.*

Proof. By 1.4.2 any two K-admissible tori can be connected by a sequence of associated K-admissible tori. Therefore it is sufficient to verify our assertion for a pair of associated tori. So let $T_1 \subset T \cdot G(a)$, $a \in \Delta$, be a K-admissible torus associated to T. We have to prove that $\varphi(T) = \varphi(T_1)$ and $\beta(T) = \beta(T_1)$. This is evidently true for φ, since $\varphi(T) \mid M \cap G(a) = \varphi(T_1) \mid M \cap G(a)$ by Lemma 5.3.1. So it remains to prove this for β.

Let $\Delta_1, ..., \Delta_s$ be those subsystems Δ_i from 5.4 which contain a. (Of course, $s \leqslant 3$.) Take $h \in G(a)(K)$ such that $T_1 = hTh^{-1}$, and take a Borel subgroup B such that $B \supset T$.

Set $B_1 = hBh^{-1}$, $G_1(b) = h^{\sigma}(B\sigma \cdot G(b))h^{-\sigma} \cap h(B \cdot G(b))h^{-1}$, $b \in \Delta$.

Then $G_1(b) = G(b)$ if $(b, a) = 0$ or $b = a$. Therefore $\beta(T, b) = \beta(T_1, b)$ if $(b, a) = 0$ or $b = a$. If $(b, a) \neq 0$, then $\Delta_i = \{b, a\}$ for some $i \in \{1, ..., s\}$. Set $\Sigma_i = \mathbf{Q}\Delta_i \cap \Sigma$. We have evidently $\langle G_1(b), G(a) \rangle = \langle G(b), G(a) \rangle = G(\Sigma_i)$. Since $\beta(T_1) \mid G(\Sigma_i) = \beta(T_1, \Sigma_i) = \beta(T, \Sigma_i) = \beta(T) \mid G(\Sigma_i)$, we see that $\beta(T_1) \mid G(\Sigma_i) = \beta(T) \mid G(\Sigma_i)$. Since subgroups $G(\Sigma_i)$, $i = 1, ..., s$, and $G(b)$, $(b, a) = 0$, $b \in \Delta$, generate G, we have $\beta(T_1) = \beta(T)$ as required.

5.5.3. *Proof of Theorem* 5.2 *concluded.* Denote $\varphi = \varphi(T)$, $\beta = \beta(T)$, where T is a K-admissible torus of G. By Lemma 5.5.2 for every K-admissible torus $T_1 \subset G$ and for every saturated connected subsystem $\tilde{\Sigma}$ of rank 2 in $\Sigma(G, T_1)$ we have

$$\alpha(g) = [\gamma(T_1, \tilde{\Sigma})](g) \cdot \beta(\varphi^0(g)) \qquad \text{for all} \quad g \in G(\tilde{\Sigma}) \cap M,$$

where $\gamma(T_1, \tilde{\Sigma})$: $M \cap G(\tilde{\Sigma}) \to C(G'(\alpha^*(\tilde{\Sigma}))) \cap M'$ is a homomorphism. Therefore

$$\alpha(g) = \beta(\varphi^0(g)) \qquad \text{for all} \quad g \in \mathscr{D}(M \cap G(\tilde{\Sigma})).$$

We recall that $M \cap G(\tilde{\Sigma}) \supseteq \mathscr{D}(\tilde{Z}_M(D))$ for an appropriate $D \in \tilde{\Pi}_3(M, T_1)$. So

$$\alpha(g) = \beta(\varphi^0(g)) \qquad \text{for} \quad g \in \mathscr{D}^2(Z_M(D)).$$

Let N be the subgroup of M, generated by all groups $\mathscr{D}^2(Z_M(D))$, where $D \in \tilde{\Pi}_3(M, T_1)$ for some K-admissible torus T_1 and $Z_G(D)$ has connected root system. Since the set of K-admissible tori is closed under conjugation by elements of $G(k)$, and, therefore, by elements of M, it follows that N is a normal subgroup of M. We have also $\alpha(g) = \beta(\varphi^0(g))$ for $g \in R$. Applying [4, Lemma 8.10] we get our theorem.

6. Main Results

Let k and k' be infinite fields. Let K be a separable quadratic extension of k. Let G be an absolutely almost simple anisotropic algebraic k-group of rank $\geqslant 2$ split by K. Let G' be a semi-simple k'-group. Let M be a big subgroup of $G(k)$ and let M' be a big subgroup of $G'(k')$. Let $\alpha: M \to M'$ be a surjective group homomorphism whose kernel is contained in the center of M.

6.1. Main Theorem. (i) *G' is k'-simple, anisotropic, and split by a quadratic separable extension of k'.*

(ii) *There exist a field homomorphism $\tilde{\varphi}$: $k \to k'$ such that $\tilde{\varphi}(k) \supseteq k'$ and $\tilde{\varphi}(k)/k'$ is separable of finite degree.*

(iii) *Let $\tilde{G}$ be the universal cover of G. Then there exists a special k'-isogeny $\tilde{\beta}$: $R_{\tilde{\varphi}(k)/k'}{}^{\tilde{\varphi}}\tilde{G} \to G'$. This special isogeny is central unless $\tilde{G}$ is of type G_2 and k is of characteristic* 3.

(iv) *Suppose that G is simply connected, or G' is adjoint, or G and G' are not of type D_{2n} and have the same root data. Then there exist a unique field homomorphism φ: $k \to \bar{k}'$ with $\varphi(k) \supseteq k'$, a unique special k'-isogeny β: $R_{\varphi(k)/k'}{}^{\varphi}G \to G'$ and a unique homomorphism γ: $M \to C(M')$ such that $\alpha(m) = \gamma(m) \cdot \beta(R^0_{\varphi(k)/k'}(\varphi^0(m)))$ for $m \in M$. The isogeny β is central unless* char $k = 3$ *and G is of type G_2. Moreover, one can take $\tilde{\varphi} = \varphi$ and β agrees with $\tilde{\beta}$ in the evident sense.*

We note that (i) *follows from* (ii) *and* (iii). *The proof of* (ii), (iii), *and* (iv) *is deduced in a standard wat* (*cf.* [4, 8.11–8.16]) *from the following*:

6.1.1. Proposition. *Theorem* 6.1(iv) *holds if G and G' are adjoint groups and G is absolutely simple.*

6.1.2. Before proving this proposition let us show that the theorem follows from it. To do this let us denote by $\bar{G}$ and $\bar{G}'$ the adjoint groups of G and G' and by $\bar{M}$ and $\bar{M}'$ the images of M and M' in $\bar{G}$ and $\bar{G}'$. Then $\bar{M}$ and $\bar{M}'$ are big subgroups of $\bar{G}$ and $\bar{G}'$ and α factors through to give an isomorphism $\bar{\alpha} = P\alpha$: $\bar{M} \to \bar{M}'$. Now $\bar{G}'$ is k-simple by Lemma 2.4.8 and by

Corollary 2.4.9 there exists a finite separable extension k'' of k' and an absolutely simple k''-group $\bar{G}''$ such that $\bar{G}' = R_{k''/k'}\bar{G}''$. We can identify $\bar{G}'(k)$ with $\bar{G}''(k'')$. Then $\bar{M}'$ becomes a big subgroup of $\bar{G}''(k'')$. This brings us into the conditions of Proposition 6.1.1. Therefore there exist a field homomorphism $\bar{\varphi}$: $k \to \bar{k}''$ and a special k''-isogeny $\bar{\beta}$: $R_{\varphi(k)/k''}{}^{\bar{\varphi}}\bar{G} \to \bar{G}''$ such that $\bar{\alpha}(\bar{m}) = \beta(R^0_{\varphi(k)/k''}(\bar{\varphi}^0(\bar{m})))$ for $\bar{m} \in \bar{M}$. Since G'' is absolutely simple, it follows that $k'' = \bar{\varphi}(k)$ so that $\bar{\beta} = R_{k''/k'}\bar{\beta}$: $R_{k''/k'}{}^{\bar{\varphi}}\bar{G} \to \bar{G}'$ is a k'-isogeny. This shows that (i) and (ii) hold always and (iii) holds if both G and G' are adjoint.

Suppose now that only G' is adjoint. Denote by ι: $G \to \bar{G}$ the canonical projection. Then we have shown that (iii) holds for $\bar{\alpha}$. Since G' is adjoint, we know that $\operatorname{Ker} \alpha = \operatorname{Ker} \iota|_M$. Thus $\alpha = \bar{\alpha} \circ \iota$. Then we have $\bar{\varphi}, \bar{\beta}$ as above. Set $\varphi = \bar{\varphi}$, $\beta = \bar{\beta} \circ \iota$. Then we see that our assertion holds in this situation too.

Suppose now that G is simply connected. Let $\varphi = \bar{\varphi}$ and let β_1: $R_{\varphi(k)/k'}{}^{\varphi}G \to \bar{G}'$ be constructed as before. Then there exists a special k'-isogeny β: $R_{\varphi(k)/k'}{}^{\varphi}G \to G'$ such that $\beta_1 = \iota' \circ \beta$, where ι': $G' \to \bar{G}'$ is the canonical projection. Set $\tilde{\alpha} = \beta \circ R^0_{\varphi(k)/k'} \circ \varphi^0$. Since $\iota' \circ \alpha = \iota' \circ \tilde{\alpha}$, it follows that $\alpha(m) \cdot \tilde{\alpha}(m)^{-1} \in C(M')$. Set $\gamma(m) = \alpha(m) \cdot \tilde{\alpha}(m^{-1})$ for $m \in M$. It follows that γ is a homomorphism γ: $M \to C(M')$. Thus we have our theorem in this case too.

Assume finally that G and G' have the same root data but are not of type D_{2n}. Then the statement follows from Lemma 5.3.4 if G is not of type G_2 and is contained in [22] if G is of type G_2.

6.1.3. *Remark.* The case D_{2n}, which is coming up all the time, could be handled as in [4, 8.11(ii)].

6.2. *Proof of Proposition* 6.1.1. By Theorem 4.1, G and G' have the same root data (since they both are adjoint). Therefore by Theorem 5.2 it is sufficient to prove Proposition 6.1.1 when G and G' are adjoint groups of rank 2 and of the same type. This is done in [22] if G is of type G_2, in [7] if G is of type A_2, in [21] if G is of type B_2 and char $k = 2$, and in 6.3 below if G is of type B_2 and char $k \neq 2$.

6.3. We assume now that G and G' are adjoint groups of type B_2 and char $k \neq 2$, char $k' \neq 2$.

THEOREM. *Proposition* 6.1.1 *holds if G and G' are both adjoint of type B_2 and* char $k \neq 2$, char $k' \neq 2$.

In the proof which will be given in several steps we identify G (resp. G') with the special orthogonal group $\mathrm{SO}(V, f)$ (resp., $\mathrm{SO}(V, f')$) of a regular anisotropic quadratic space of dimension 5 over k (resp. k') with quadratic form f (resp. f'). We denote by F (resp., F') the bilinear form corresponding to f' (resp. F') and by $N^{\perp}$ the orthogonal complement of $N \subseteq V$. For a subspace $N \subseteq V$ we consider projectivisation PN of N as lying in PV. Next $\tilde{G}_N = \{x \in G \mid xN = N\}$. The group $\tilde{G}_N$ may not be connected bu $G_N = \tilde{G}_N^0$ has index at most two in $\tilde{G}_N$. Therefore $h^2 \in G_N$ for $h \in \tilde{G}_N$. This property together with the property 2.3.2(iii) of big subgroups permits us to characterize the groups G_N.

Let $\hat{\Pi}_2(G, T) = \{\tilde{T} \in \tilde{\Pi}_2(G, T) \mid Z_G(\mathscr{D}^i(Z_G(\tilde{T}))^0$ is commutative for any $i \geqslant 1\}$. It means that $\Sigma(\tilde{T})$ consists of two opposite short roots. Let $\hat{\Pi}_2(G) = \bigcup_{T \text{ maximal}} \hat{\Pi}_2(G, T)$. Introduce in $\hat{\Pi}_2(G)$ an incidence structure. The elements of $\hat{\Pi}_2(G)$ are called points and $T_1, T_2 \in \hat{\Pi}_2(G)$ are called associated if $(Z_G(\mathscr{D}^i(\langle T_1, T_2 \rangle)))^0 \in \hat{\Pi}_2(G)$ and $\mathscr{D}^i(\langle T_1, T_2 \rangle) \neq 1$ for all $i \geqslant 1$.

6.3.1. PROPOSITION. *This incidence structure makes $\hat{\Pi}_2(G)$ into a Grassmanian of lines in a four-dimensional projective space.*

This result, Chow's theorem [6, III, Sect. 2], and the fundamental theorem of projective geometry easily imply that an abstract isomorphism of big subgroups of adjoint groups of type B_2 is good.

6.3.2. LEMMA. *Let N be a plane of V, $T \in \hat{\Pi}_2(G)$. Then*

(i) $G_{N^\perp} = G_N$;

(ii) *there exists a unique $T(N) \in \hat{\Pi}_2(G)$ such that $G_N = Z_G(T(N))$;*

(iii) *there exists a unique plane $N(T)$ such that $T(N(T)) = T$; in this case T acts trivially on $N(T)^\perp$.*

6.3.3. LEMMA. *Let N_1, N_2 be planes in V. Then* $\dim(N_1 \cap N_2) = 1$ *if and only if both of the following conditions are satisfied:*

(i) $\mathscr{D}^i\langle T(N_1), T(N_2)\rangle \neq 1$ *for all* $i \geqslant 1$,

(ii) $T = Z_G(\mathscr{D}^i(\langle T(N_1), T(N_2)\rangle))^0 \in \hat{\Pi}_2(G)$ *for all* $i \geqslant 1$.

In this case T acts trvially on $N_1 + N_2$.

Proof. Set $W = N_1 + N_2$, $H = \langle T(N_1), T(N_2)\rangle$. Suppose first that $\dim(N_1 \cap N_2) = 1$. Then $\dim W = 3$ and $\dim W^\perp = 2$. Since both $T(N_i)$ act trivially on $W^\perp$, it follows that H acts trivially on $W^\perp$, whence $T(W^\perp) \subset Z(\mathscr{D}^i H)$ for all $i \geqslant 0$. Clearly, $H = \mathrm{SO}(W)$ whence $\mathscr{D}^i H = H$ and $Z_G(H)^0 = T(W^\perp)$, as required.

Now suppose that N_1, N_2 are such that $T = Z_G(\mathscr{D}^i(H) \in \hat{\Pi}(G)$ for all $i \geqslant 1$ and H is not solvable. Since T acts irreducibly on a two-dimensional space $N = N(T)$ and trivially on $N^\perp$ it follows that either $N_i \subset N^\perp$ or $N_i = N$. If one of N_i, say N_1, coincides with N, then automatically $[N_1, N_2] = 1$, a contradiction with (i). So $W \subset N^\perp$. If $\dim W = 2$, then $N_1 = N_2$, i.e., $T(N_1) = T(N_2)$, a contradiction with (i). So $\dim W = 3$ as required.

6.3.4. *Remark.* Proposition 6.3.1 follows from Lemma 6.3.3.

6.3.5. Let M be a big subgroup of $G(k)$. Let $\hat{\Pi}_2(M, T)$ be the set of $D \in \tilde{\Pi}_2(M, T)$ such that the subgroup of $Z_M(\mathscr{D}^i Z_M(D))$ generated by squares is commutative for all $i \geqslant 1$. Let $\hat{\Pi}_2(M)$ be the union of $\hat{\Pi}_2(M, T)$ over all maximal k-tori T of G. Let us say that $D_1, D_2 \in \hat{\Pi}_2(M)$ are associated if $Z_M(\mathscr{D}^i(\langle D_1, D_2\rangle)) \supset D \in \hat{\Pi}_2(H)$ and $\mathscr{D}^i\langle D_1, D_2\rangle \neq 1$ for all $i \geqslant 1$.

Lemma. *The bijection $\tilde{\Pi}_2(G) \to \tilde{\Pi}_2(H)$ of* 3.1.4. *induces a bijection $\hat{\Pi}_2(G) \to \hat{\Pi}_2(H)$ which is an isomorphism of the corresponding incidence structures.*

Proof. Direct application of properties of big subgroups.

6.3.6. Let now M' be a big subgroup of $G'(k')$ and let $\alpha: M \to M'$ be a group isomorphism.

Lemma. *There exists a unique isomorphism $\varphi: k \to k'$ and a unique (up to multiplication by a constant) φ-semilinear orthogonality-preserving isomorphism $\varphi: V \to V'$ such that $\alpha(h) = P\mu \circ h \circ P\mu^{-1}$ for $h \in M$.*

Proof. The induced mapping α_*: $\hat{\Pi}_2(G) \to \hat{\Pi}_2(G')$ is (by Proposition 6.3.1 and 6.3.5) an isomorphism of incidence structures. By Chow's theorem [6, III, Sect. 2] α_* is induced by a semi-linear (with respect to appropriate $\varphi: k \to k'$) isomorphism $\mu: V \to V'$. The fact that μ preserves orthogonality follows from Lemma 6.3.2(i).

6.3.7. Corollary. *Theorem* 6.3 *holds.*

Proof. The first assertion is evident. In case (ii) take $T(N) = C(G_N)$. Then $T(N)$ acts irreducibly on N and trivially on $N^\perp$. Thus $\mathscr{D}Z_G(T(N)) = \mathrm{SO}(N^\perp, f \mid N^\perp)$. Since $\dim N^\perp = 3$, this group is of type A_1 whence $Z_G(\mathscr{D}Z_G(T(N)))^0 = \mathrm{SO}(N, f \mid N)$. Thus $T(N) \in \tilde{\Pi}_2(G)$. The other property is clear. Let us prove (iii). Since $\mathscr{D}Z_G(T)$ is semi-simple and corresponds to short roots, it follows that $\mathscr{D}Z_G(T) = \mathrm{SO}(\tilde{N}, f \mid \tilde{N})$ for some $\tilde{N}$ with $\dim \tilde{N} = 3$. Therefore T acts trivially on $\tilde{N}$. Set $N(T) = \tilde{N}^\perp$. Now (iii) is clear.

6.3.8. *Remark.* When *I* almost completed writing the present version of the paper I was certain thatTheorem 6.3 is contained in the literature. However it turned out to be not so, and therefore I was forced to add Section 6.3.

7. Open Questions

Let k and k' be infinite fields, let G be an absolutely almost simple algebraic k-group, and let G' be an absolutely almost simple algebraic k'-

group. Let M be a dense subgroup of G and let α: $M \to G'$ be a homomorphism with dense image.

7.1. The following questions are suggested by [7, 20–23] and the present paper.

7.1.1. Extend the results of the present paper to anisotropic groups isotropic (but not necessarily split) over a separable quadratic extensions.

Note that the association (cf. 1.4) still exists, and analogs of the results of Section 1 hold. However the methods of Section 4 are too crude to give an easy extension. Another difficulty is that in Section 5 we would get a reduction to groups of relative rank 2 over a quadratic extension. It would be quite a problem to handle them case by case as we did in [7, 21, 22]. It seems that an analog of the Borel–Tits maximal split subgroup [2, Sect. 7; 3, Sect. 4] is needed to reduce the problem to groups of rank 2 split over a quadratic extension.

7.1.2. Extend the result of the present paper to groups which have a maximal k-torus split over a cyclic Galois extension of prime degree $p > 2$.

Some results about these groups are contained in [19, no. 18]. But the theorem about association (cf. 1.4) is not proved there.

7.1.3. The method used in [7] to study monomorphisms between unitary groups may be extendable to other groups. For example, one could try to handle anisotropic forms of C_n and D_n connected with division algebras with involution of the first kind. Of course, anisotropic forms of C_n connected with quaternions are split over a quadratic extension. Thus the present paper gives at least partial result (with M and $\alpha(M)$ big). But one could try to weaken the assumptions on M and on $\alpha(M)$ to put them in line with assumptions of [7, 13, 21, 22].

7.1.4. It also seems that "exceptional" groups of type D_4 (which are connected with cubic extensions) are in the same position to groups of type F_4 as unitary groups (of type A_2) are to G_2. Thus one may try to extend the result of [7] to groups of type D_4 connected with separable three-dimensional algebras K/k, and then to apply as in [22] the obtained result to find all monomorphisms between groups of type F_4 (which are the automorphism groups of exceptional Jordan algebras).

7.1.5. More generally, the method of [22] (which consisted of setting up a geometry on the set of subalgebras) may be applicable to different groups of the form Aut A where A is an algebra.

7.2. The questions about homomorphism between algebraic groups are connected with some geometric questions, and, often, are equivalent to them. For example,

(i) Almost all the literature on the automorphisms and isomorphisms started by O. Schreier and B. L. van der Waerden and carried through 50 years by J. Dieudonné, Hua Lo Keng, and O. T. O'Meara and his school) is based on the Fundamental Theorem of Projective Geometry (abbreviated FTPG).

(ii) The result of A. Borel and J. Tits implies (and, probably, is equivalent to) a generalization of FTPG to Tits buildings (of spherical type).

(iii) The result of G. Mostow and G. Margulis is based on an (ingeneous) adaptation of the result about isomorphisms of Tits' buildings.

(iv) The results of B. R. MacDonald [9] are based on FTPG over local rings.

(v) The result of [22] is based on some geometry of the set of subalgebras of an algebra of Cayley numbers.

(vi) The result of [23] is based on a particular case of the Klingenberg's [8] generalization of FTPG.

(vii) In a recent paper [24] D. James used a generalization of Klingenberg's result to study homomorphisms between full orthogonal groups.

(viii) In the present paper we also use the geometry of association of K-admissible tori.

7.2.2. The results of [7, 22] suggest that behind the method of these papers (as well as behind the methods of the present paper and some papers of O'Meara and his school) there is a geometry (or geometries) which are different from Tits geometries. For example the geometry used in [7] actually is a geometry on the set of tori. A variant of this geometry has been used by Ruth M. Charney and Karen Vogtman to study homology stability of some classical groups over rings. In a way reminiscent of the Tits buildings of an isotropic group the geometries which occur in our cases can be defined, for example, in one of the following ways.

(i) Let G be an algebraic group and let V be the set of all tori T of G such that $T = C(Z_G(T)^0$ with the natural order relation.

(ii) Let G, V be as above. For every subgroup H of G (preferably containing a maximal torus) we denote by V_H the set of tori from V contained in H. Then the set of subsets V_H when H runs through subgroups of G defines a geometry on V. The set V carries a structure of a union of affine algebraic varieties. One can try to compactify it. One way to compactify is to identify elements of V with their Lie algebras and then compactify in the corresponding Grassmanian (cf. [7]). Another way to compactify was suggested by D. Kazhdan. Consider the fibering $G \to G/B$ where B is a Borel subgroup. Then represent the fiber as $T \times U$, compactify

T to obtain $\hat{T}$ (for example, as D. Mumford in "Toroidal embeddings") and compactify U to obtain G/B. Let $\hat{G}$ be the associated fibering over GB with fiber $\hat{T} \times G/B$. Now $\hat{G}$ is complete and we can complete V in the Chow scheme of $\hat{G}$.

7.2.3. Another interesting question is to study not only isomorphisms between geometries but also homomorphisms and not everywhere defined maps. This was done in different situations by Klingenberg [8] for projective spaces and Margulis [10] for Tits buildings. It seems interesting to extend Klingenberg's result on (at least everywhere defined) homomorphisms to Tits buildings. Such results may lead to new formulations of theorems of homomorphisms between subgroups of algebraic groups.

7.3. We will now discuss the situation with homomorphisms, i.e., what happens if one does not assume (implicitly or explicitly) that $\operatorname{Ker} \alpha$ is finite. Margulis [10] has not imposed that assumption but his α are automatically almost monomorphisms (the kernel is finite). A case where $\operatorname{Ker} \alpha$ is non-trivial even poset priori and the structure of normal subgroups is not known a priori is [23]. The result there says that G carries a structure of a group scheme over some ring and α is roughly a homomorphism of reduction modulo some ideal of that ring. Another example of this kind is a recent result of D. James on homomorphisms of full orthogonal groups [24].

7.3.1. As we indicated in [23, 0.8] a general statement for a big (or full) in an appropriate sense subgroup M should be:

There exist a subring $A \subseteq k$ with the quotient field k; a structure G_A of a group scheme over A on G such that $M \subseteq G_A(A)$; a homomorphism or rings φ: $A \to k$, a special k'-isogeny β: ${}^{\varphi}G_A \to G'$ and a homomorphism γ: $M \to C(G'(k'))$ such that $\alpha(m) = \gamma(m) \cdot \beta(\varphi^0(m))$ for $m \in M$.

It seems that Klingenberg's result should give a result of this sort when G and G' are of the type A_n, H is a full subgroup of G, and the image under α of the group generated by transvections belonging to H is dense in G'.

However, one could not hope to drop too many assumptions on α and M. For example, if M is free, then it can be easily mapped anywhere. The group schemes which would come up may be not semi-simple since one can easily construct (full) integral subgroups of split groups which are mapped by reduction into groups of smaller rank or even of the same rank but of different type (as is the case with most parahoric subgroups of algebraic groups over local fields). So one may need an assumption that G' is not "too small."

If a result about homomorphisms of Tits buildings were available, one would be able to use it to study homomorphisms of full subgroups of isotropic algebraic groups. In particular, it seems possible to study monomorphisms of full subgroups of isotropic groups using the known results about the maps between Tits buildings.

7.3.2. Another method to study homomorphisms would be through generators and relations. Those are available for Chevalley groups over most rings.

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